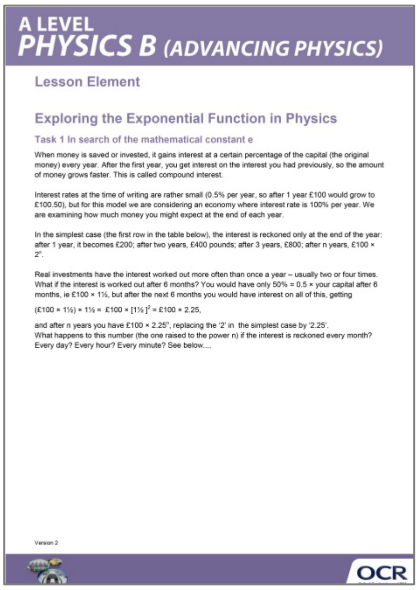
Lesson Element

Exploring the Exponential Function in Physics

Instructions and answers for teachers

These instructions should accompany the OCR resource '*Exploring the Exponential Function in Physics *activity' *which supports OCR A Level Physics B.*


A LEVEL
PHYSICS B (ADVANCING PHYSICS)

Lesson Element

Exploring the Exponential Function in Physics

Task 1 In search of the mathematical constant e

When money is saved or invested, it gains interest at a certain percentage of the capital (the original money) every year. After the first year, you get interest on the interest you had previously, so the amount of money grows faster. This is called compound interest.

Interest rates at the time of writing are rather small (0.5% per year, so after 1 year £100 would grow to £100.50), but for this model we are considering an economy where interest rate is 100% per year. We are examining how much money you might expect at the end of each year.

In the simplest case (the first row in the table below), the interest is reckoned only at the end of the year: after 1 year, it becomes £200; after two years, £400 pounds; after 3 years, £800; after n years, £100 × 2^n.

Real investments have the interest worked out more often than once a year – usually two or four times. What if the interest is worked out after 6 months? You would have only 50% = 0.5 × your capital after 6 months, ie £100 × 1½, but after the next 6 months you would have interest on all of this, getting

(£100 × 1½) × 1½ = £100 × $[1½]^2$ = £100 × 2.25,

and after n years you have £100 × 2.25^n, replacing the '2' in the simplest case by '2.25'.
What happens to this number (the one raised to the power n) if the interest is reckoned every month? Every day? Every hour? Every minute? See below....

Version 2

The Activity:

This activity offers an opportunity for English skills development.

This activity offers an opportunity for maths skills development.

Associated materials:

'Exploring the Exponential Function in Physics' activity.

In this article a range of practical and theoretical activities are suggested to allow students to aim towards understanding the following key features of exponentials. It is assumed here that all changes are with respect to time, but students should be able to transfer the idea to eg penetration of gamma radiation through different depths of a medium of uniform density.

The following points are stages in understanding of exponentials at this level, with the difficulty generally increasing down the list.

1. Changes of any quantity where the change is directly proportional to the quantity itself result in an exponential variation of that quantity

2. exponential changes can be growth or, more commonly, decay, depending on whether the change is an increase or a decrease;

3. the ratio of values of the varying quantity with fixed time intervals is constant – this is often restricted to the ratio of ½, but students should recognise that any value could be used, and some other values are more relevant (such as 37% in capacitor discharge);

4. the rate of change per unit time of the quantity *Y* can be represented as

   $\frac{\Delta Y}{\Delta t}$ for larger changes, but at the limit becomes $\frac{\mathrm{d}Y}{\mathrm{d}t}$ which is the instantaneous gradient of the *Y*-*t* graph at that point;

5. in the case of decay (the commonest application in physics), dependence of this rate of change on the quantity itself is given by $\frac{\mathrm{d}Y}{\mathrm{d}t} = -KY$

   where *K* is a constant (the decay constant);

6. the two physical contexts where exponential decay is studied are radioactive decay and discharging of a capacitor;

7. the solution to the equation given in 5 is

   $Y = Y_0 \mathrm{e}^{-Kt} = \frac{Y_0}{\mathrm{e}^{Kt}}$ where the mathematical constant e = 2.718281..... ;

8. data following an exponential variation, such as *Y* above, can be converted into a linear graph by plotting a graph of ln *Y* against *t*; the gradient of the resulting graph is then – *K* [as ln (e^{-Kt}) = *Kt*, giving ln *Y* = ln Y_0 – *Kt* which corresponds to *y* = *c* + *mx*].

Although students studying A Level maths will probably have covered first-order differential equations, and the formal solution to $\frac{\mathrm{d}Y}{\mathrm{d}t} = -KY$ is quick and elegant, other students will not have access to this; furthermore, few A level mathematicians have a firm grasp of the physical significance of the exponential function.

It is also essential that all candidates experience the iterative modelling approach where the formal mathematical method is inaccessible; this approach is expected as preparation for the examinations.

It is important to recognise the fact that, at GCSE, most candidates have a very tentative grasp of the idea of radioactive half-life; GCSE Principal Examiners usually report that finding a count rate after a whole number of half-lives, such as three or four, is a skill associated only with A* students. Furthermore, GCE AS and A2 examiners frequently report that candidates use the word ‘exponentially’ to mean ‘non-linearly’, whereby any increase or decrease which is faster than linear is described as exponential.

Suggested activities

Much will depend on the degree of mathematical competence of the students. Familiarity with the notation of differential calculus is important, but competent sixth-form mathematicians often have an imperfect grasp of the physics involved in the equations being used. A range of activities to allow students of different mathematical aptitudes access to this important area is suggested here.

Teachers will want to vary their approach depending on their group, and on their own preferences: some would prefer to start with experimental work, while others would prefer to develop the theory first.

A suggested practical activity would be a simple experiment measuring the head on a frothy drink over time. This could provide data which can be analysed to see whether it follows exponential behaviour or not. An example experiment is available on the Advancing Physics website at http://fdslive.oup.com/www.oup.com/oxed/secondary/science/advancingphysics/support_materials/a2_teacher/a2_chapter_10.html

The advancing Physics CD has many useful resources. Activity 160S illustrates the idea of an iterative solution and introduces the number e.

The activities presented here are provided to give alternatives from which to choose. Other teaching methods may also be introduced: teachers may well wish to use the iconic modelling program Worldmaker, which is akin to rolling many dice to simulate decay.

The outcomes from the suggested activities here (which could be up to 2 lessons in length are) are:

- understanding how to test experimental data for exponential behaviour, including generalising the idea of 'half-life' to include any constant ratio and (with classes ready for that development at this state) use of the relationship between exponential and logarithmic functions
- using an iterative model to see how the differential equation actually produces the correct outcome without any mathematics other than simple calculation and re-calculation
- developing an understanding of the first-order differential equation which produces exponential decay, and how the parameters of the equation effect the outcome
- realising that e is just a number which arises naturally from the basic idea of the rate of increase (or decrease) of a quantity being proportional to the amount present.

Task 1 In search of the mathematical constant e

An iterative approach, based on the idea of compound interest, was constructed to show how an exponential equation involving e arises naturally from the idea of exponential changes with increasingly small time intervals.

Task 2a Using Modellus to model radioactive decay

The use of the software *Modellus* to model radioactive decay. This is a very powerful software modelling program which is strongly recommended. *Modellus* is supplied on the Advancing Physics CDs, but it can also be downloaded from http://modellus.fct.unl.pt/. The student will quickly enter the basic equation, whereupon the software will plot graphs and create tables of the solution, using the boundary conditions entered by the student.

Note that 'the probability of any nucleus decaying in the time interval chosen' does not imply 'the probability of any nucleus decaying in one second' as, for example, the half-life of radon-213 is 0.025 s, giving $\lambda = 27.7\ s^{-1}$; clearly a probability of >1 is not feasible, but this can be interpreted as 0.0277 ms^{-1}, or a probability of about 1 in 36 of decaying in a millisecond.

Task 2b Using Excel to model radioactive decay

Microsoft Excel can also be used to model radioactive decay, in exactly the same way as *Modellus*, although it is less elegant. This is an alternative to using Modellus (task 2a) for teachers who prefer Excel, or whose students are likely to be familiar with Excel. The instructions have been written for the absolute Excel novice as many users become competent at doing certain tasks in Excel without necessarily understanding what's going on.

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