OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4758/01**

MATHEMATICS (MEI)

Differential Equations

MONDAY 18 JUNE 2007

Morning
Time: 1 hour 30 minutes

Additional materials:
- Answer booklet (8 pages)
- Graph paper
- MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- There is an **insert** for use in Question **3**.

ADVICE TO CANDIDATES

- Read each question carefully and make sure that you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages and an insert.

  [Turn over

1 An object is suspended from one end of a vertical spring in a resistive medium. The other end of the spring is made to oscillate and the differential equation describing the motion of the object is

$$\ddot{y} + 4\dot{y} + 29y = 3\cos t,$$

where y is the displacement at time t of the object from its equilibrium position.

(i) Find the general solution. [11]

(ii) Find the particular solution subject to the conditions $\dot{y} = y = 0$ when $t = 0$. What is the amplitude of the motion for large values of t? [8]

(iii) Find the displacement and velocity of the object when $t = 10\pi$. [2]

At $t = 10\pi$, the upper end of the spring stops oscillating and the differential equation describing the motion of the object is now

$$\ddot{y} + 4\dot{y} + 29y = 0.$$

(iv) Write down the general solution. Describe briefly the motion for $t > 10\pi$. [3]

2 The differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 1 + x^n,$$

where n is a positive constant, is to be solved for $x > 0$.

First suppose that $n \neq 2$.

(i) Find the general solution for y in terms of x. [8]

(ii) Use your general solution to find the limit of y as $x \to 0$. Show how the value of this limit can be deduced from the differential equation, provided that $\frac{\mathrm{d}y}{\mathrm{d}x}$ tends to a finite limit as $x \to 0$. [3]

(iii) Find the particular solution given that $y = -\frac{1}{2}$ when $x = 1$. Sketch a graph of the solution in the case $n = 1$. [4]

Now consider the case $n = 2$.

(iv) Find y in terms of x, given that y has the same value at $x = 1$ as at $x = 2$. [9]

3 There is an insert for use with part (iii) of this question.

Water is draining from a tank. The depth of water in the tank is initially 1 m, and after t minutes the depth is y m.

The depth is first modelled by the differential equation

$$\frac{dy}{dt} = -k\sqrt{y}\,(1 + 0.1\cos 25t),$$

where k is a constant.

(i) Find y in terms of t and k. [8]

(ii) If the depth of water is 0.5 m after 1 minute, show that $k = 0.586$ correct to three significant figures. Hence calculate the depth after 2 minutes. [4]

An alternative model is proposed, giving the differential equation

$$\frac{dy}{dt} = -0.586\left(\sqrt{y} + 0.1\cos 25t\right). \qquad (*)$$

The insert shows a tangent field for this differential equation.

(iii) Sketch the solution curve starting at $(0, 1)$ and hence estimate the time for the tank to empty. [4]

Euler's method is now used. The algorithm is given by $t_{r+1} = t_r + h$, $y_{r+1} = y_r + h\dot{y}_r$, where $\dot{y}$ is given by (*).

(iv) Using a step length of 0.1, verify that this gives an estimate of $y = 0.935\,54$ when $t = 0.1$ for the solution through $(0, 1)$ and calculate an estimate for y when $t = 0.2$. [6]

(v) Use (*) to show that when the depth of water is less than 1 cm the model predicts that $\frac{dy}{dt}$ is positive for some values of t. [2]

[Question 4 is printed overleaf.]

4 The following simultaneous differential equations are to be solved.

$$\frac{dx}{dt} = -5x + 4y + e^{-2t},$$
$$\frac{dy}{dt} = -9x + 7y + 3e^{-2t}.$$

(i) Show that $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = 3e^{-2t}$. [5]

(ii) Find the general solution for x in terms of t. [8]

(iii) Hence obtain the corresponding general solution for y, simplifying your answer. [4]

(iv) Given that $x = y = 0$ when $t = 0$, find the particular solutions. Find the values of $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ when $t = 0$. Sketch graphs of the solutions. [7]