ADVANCED GCE

4758/01

MATHEMATICS (MEI)

Differential Equations

THURSDAY 24 JANUARY 2008

Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):
Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

[Turn over

1 The differential equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = f(t)$ is to be solved for $t \geqslant 0$ subject to the conditions that $\frac{dy}{dt} = 0$ and $y = 0$ when $t = 0$.

Firstly consider the case $f(t) = 2$.

(i) Find the solution for y in terms of t. [10]

Now consider the case $f(t) = e^{-t}$.

(ii) Explain briefly why a particular integral cannot be of the form ae^{-t} or ate^{-t}. Find a particular integral and hence solve the differential equation, subject to the given conditions. [8]

(iii) For $t > 0$, show that $y > 0$ and find the maximum value of y. Hence sketch the solution for $t \geqslant 0$. [You may assume that $t^k e^{-t} \to 0$ as $t \to \infty$ for any k.] [6]

2 A raindrop falls from rest through mist. Its velocity, $v\,\mathrm{m\,s^{-1}}$ vertically downwards, at time t seconds after it starts to fall is modelled by the differential equation

$$(1+t)\frac{dv}{dt} + 3v = (1+t)g - 3.$$

(i) Solve the differential equation to show that $v = \frac{1}{4}g(1+t) - 1 + (1 - \frac{1}{4}g)(1+t)^{-3}$. [10]

The model is refined and the term -3 is replaced by the term $-2v$, giving the differential equation

$$(1+t)\frac{dv}{dt} + 3v = (1+t)g - 2v.$$

(ii) Find the solution subject to the same initial conditions as before. [9]

(iii) For each model, describe what happens to the acceleration of the raindrop as $t \to \infty$. [5]

3 The population, P, of a species at time t years is to be modelled by a differential equation. The initial population is 2000.

At first the model $\frac{dP}{dt} = 0.5P$ is used.

(i) Find P in terms of t. [3]

To take account of observed fluctuations, the model is refined to give $\frac{dP}{dt} = 0.5P + 170 \sin 2t$.

(ii) State the complementary function for this differential equation. Find a particular integral and hence state the general solution. [8]

(iii) Find the solution subject to the given initial condition. [2]

The model is further refined to give $\frac{dP}{dt} = 0.5P + P^{\frac{2}{3}} \sin 2t$. This is to be solved using Euler's method. The algorithm is given by $t_{r+1} = t_r + h$, $P_{r+1} = P_r + h\dot{P}_r$.

(iv) Using a step length of 0.1 and the given initial conditions, perform two iterations of the algorithm to estimate the population when $t = 0.2$. [4]

The population is observed to tend to a non-zero finite limit as $t \to \infty$, so a further model is proposed, given by

$$\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{12\,000}\right)^{\frac{1}{2}}.$$

(v) Without solving the differential equation,

(A) find the limiting value of P as $t \to \infty$, [3]

(B) find the value of P for which the rate of population growth is greatest. [4]

4 The simultaneous differential equations

$$\frac{dx}{dt} = -3x + y + 9,$$
$$\frac{dy}{dt} = -5x + y + 15,$$

are to be solved for $t \geqslant 0$.

(i) Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 6$. [5]

(ii) Find the general solution for x. [7]

(iii) Hence find the corresponding general solution for y. [3]

(iv) Find the solutions subject to the conditions that $x = y = 0$ when $t = 0$. [4]

(v) Sketch, on separate axes, graphs of the solutions for $t \geqslant 0$. [5]