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LEVEL 3 CERTIFICATE MATHEMATICS FOR ENGINEERING

H860/02 Paper 2

PRE-RELEASE MATERIAL

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Analysis of RLC circuits

An RLC circuit is an electrically connected arrangement of resistors, inductors and capacitors. The arrangement and values of these and other components will determine the function of the circuit. Such circuits can be found in equipment ranging from simple dimmer switches to radios and advanced navigation systems. This document describes the basic characteristics of resistors, inductors and capacitors together with a mathematical analysis of circuits with a DC voltage source and switch.

Component units

Each component has a value expressed in an associated unit. For a resistor the unit of resistance is the ohm (Ω), for an inductor the unit of inductance is the henry (H) and for a capacitor the unit of capacitance is the farad (F). The standard convention for expressing multiples of units is used; for example
$2\,\text{k}\Omega = 2\times10^3\,\Omega$, $30\,\text{mH} = 30\times10^{-3}\,\text{H}$, $5\,\mu\text{F} = 5\times10^{-6}\,\text{F}$, $10\,\text{nH} = 10\times10^{-9}\,\text{H}$ and $50\,\text{pH} = 50\times10^{-12}\,\text{H}$.

When drawing electrical circuits, standard symbols are used for each component. An example of a circuit diagram containing symbols for a resistor, an inductor, a capacitor, a single-pole two-way switch and a DC voltage source is shown in Fig. 1.

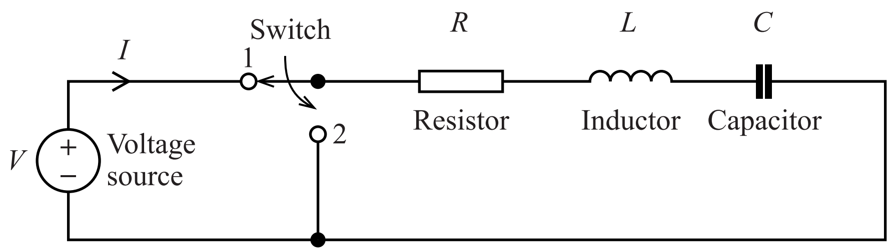


Fig. 1

Resistors

A resistor is a component that restricts the flow of electrical current. When a voltage of V (volts) is applied across a resistor of resistance R (ohms), a current of I (amperes) will flow though the resistor. According to Ohm's law

$$I = \frac{V}{R}.$$

Fig. 2 shows a circuit comprising a resistor of resistance R, together with an applied DC voltage, V, and a current, I.

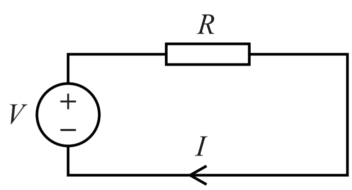
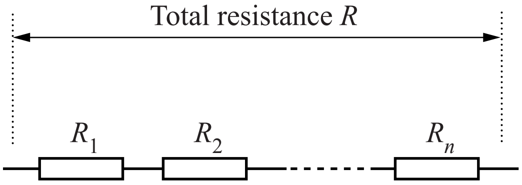


Fig. 2

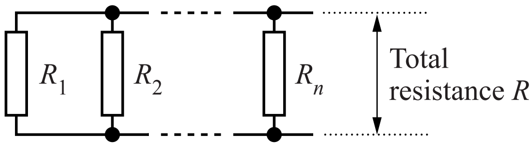
Resistors can be connected in series, in parallel or in a combination of both, as shown in Fig. 3a, 3b and 3c respectively.



Resistors connected in series

$R = R_1 + R_2 + \cdots + R_n$

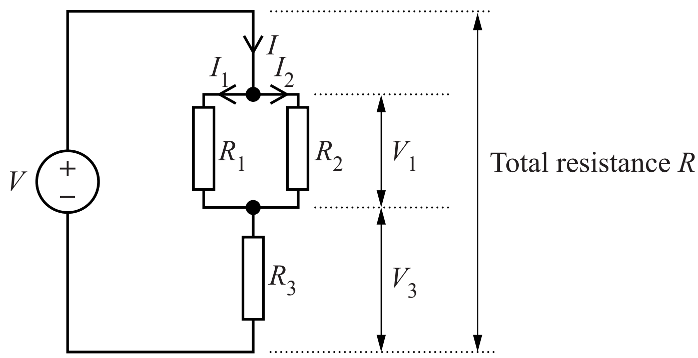
Fig. 3a



Resistors connected in parallel

$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$

Fig. 3b



Resistors connected in parallel and series

$$R = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1 + R_2}$$

Fig. 3c

The total resistance, R, of a group of n resistors connected in series, as shown in Fig. 3a, is

$$R = R_1 + R_2 + \cdots + R_n.$$

For a group of n resistors connected in parallel, as shown in Fig. 3b, the relationship between the total resistance, R, and the individual resistors is

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}\,.$$

By combining these relationships the total resistance of a collection of connected resistors can be established. The total resistance, R, of the circuit shown in Fig. 3c is

$$R = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} + R_3 = \frac{R_1R_2}{R_1 + R_2} + R_3 = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1 + R_2}\,.$$

When resistors are connected together in a circuit it is often necessary to calculate the current flowing through each resistor and the potential difference (i.e. the voltage drop) across each resistor. These calculations can be made by applying Kirchhoff's current law and/or Kirchhoff's voltage law.

Kirchhoff's current law states
the algebraic sum of the currents flowing into and out of any junction in a circuit is zero.

Kirchhoff's voltage law states
the algebraic sum of the potential differences across each component (including voltage sources) in a circuit loop is zero.

The total current, I, in the circuit shown in Fig. 3c is

$$I = \frac{V}{R} = V\left(\frac{R_1 + R_2}{R_1R_2 + R_1R_3 + R_2R_3}\right).$$

The potential difference, V_3, across R_3 is

$$V_3 = IR_3 = V\left(\frac{R_1 + R_2}{R_1R_2 + R_1R_3 + R_2R_3}\right)R_3.$$

By applying Kirchhoff's voltage law, the potential difference, V_1, across the two resistors R_1 and R_2 connected in parallel is

$$V_1 = V - V_3 = V\left(1 - \frac{(R_1 + R_2)R_3}{R_1R_2 + R_1R_3 + R_2R_3}\right).$$

From this the currents I_1 and I_2 are

$$I_1 = \frac{V_1}{R_1}, \qquad I_2 = \frac{V_1}{R_2}.$$

Note that $I = I_1 + I_2$, in accordance with Kirchhoff's current law.

Capacitors

A capacitor is a component that consists of two conducting surfaces separated by a non-conducting material. A capacitor can store electrical energy in an electric field between the two surfaces. Fig. 4 shows a circuit comprising a capacitor, a resistor, a single-pole two-way switch and a DC voltage source.

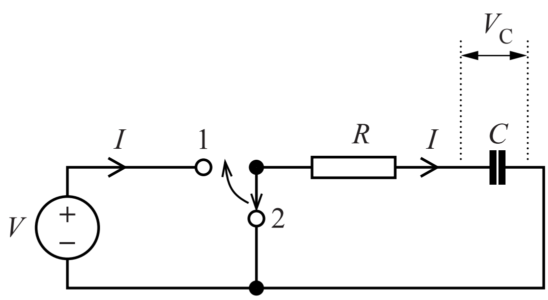


Fig. 4

Initially the switch is in position 2 and the capacitor has no stored electrical energy. When the switch is moved to position 1 there will be a flow of current, I, for a short time around the circuit until the potential difference across the capacitor, V_C , becomes the same as the supplied voltage, V. At this time the capacitor is fully charged with $V_C = V$ and $I = 0$. When the switch is then moved back to position 2, the capacitor will begin to discharge and there will be a flow of current through the resistor for a short time in the opposite direction until the capacitor has become completely discharged with $V_C = 0$ and $I = 0$.

The relationship between the current, I, flowing through the capacitor, and the potential difference, V_C, can be modelled by the differential equation

$$I = C\frac{\mathrm{d}V_C}{\mathrm{d}t}$$

where t is time measured in seconds and C is the capacitance in farads.
An implication of this relationship is that when the potential difference across a capacitor remains constant, as in the case of a fully charged or discharged capacitor, no current flows.

In the circuit shown in Fig. 4 the capacitor is being charged when the switch is in position 1. Kirchhoff's voltage law can be used to provide the equation

$$V - V_C = RI.$$

By substituting $I = C\frac{\mathrm{d}V_C}{\mathrm{d}t}$ and rearranging the result, the following differential equation can be established.

$$\frac{\mathrm{d}V_C}{\mathrm{d}t} + \frac{V_C}{RC} = \frac{V}{RC}$$

Provided that V remains constant the general solution to this differential equation is

$$V_C = V - K\mathrm{e}^{-\frac{t}{RC}}$$

where K is a constant and t is the time after the capacitor begins to be charged.

When the capacitor is fully charged and the switch is moved to position 2, the capacitor discharges and

$$V_C = K\mathrm{e}^{-\frac{t}{RC}}$$

where t is the time after the capacitor begins to be discharged.

By considering initial conditions, the constant K can be evaluated to give separate particular solutions for the cases of a charging and a discharging capacitor.

Inductors

An inductor is a component that consists of a conducting wire wound into a coil. An inductor stores energy in a magnetic field when a current passes through it. In an ideal inductor the potential difference, V_L, across it is proportional to the rate of change of current, I, and can be modelled by the differential equation

$$V_L = L\frac{dI}{dt}$$

where t is time measured in seconds and L is the inductance in henry.
An implication of this relationship is that when a constant current flows through an inductor, the potential difference across it is zero.

Fig. 5 shows a circuit comprising an inductor, a resistor, a single-pole two-way switch and a DC voltage source.

From Kirchhoff's voltage law

$$V - V_L = RI.$$

Substituting $V_L = L\frac{dI}{dt}$ gives

$$V - L\frac{dI}{dt} = RI,$$

rearranging gives the differential equation

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{V}{L}.$$

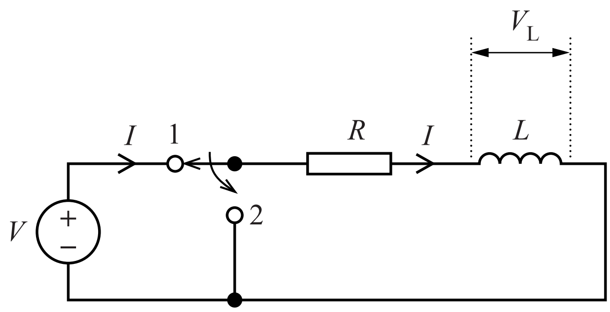


Fig. 5

This equation may be solved to provide a general solution that relates the current I to the applied voltage V, time t, and the constant values R and L. Particular solutions may then be found by considering initial conditions.

A closed LC circuit

Fig. 6 shows a circuit containing a capacitor, a single-pole two-way switch, an inductor, a resistor and a DC voltage source.

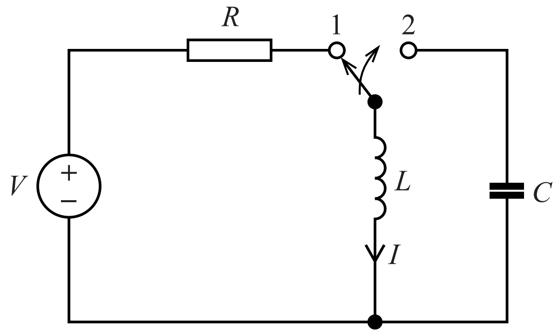


Fig. 6

When the switch is in position 1, a current will flow through the resistor and inductor. The flow of current will cause a magnetic field to form around the inductor so storing electrical energy. When the switch is then moved to position 2, the magnetic field will begin to collapse and the capacitor starts to charge. As the current diminishes to zero, the capacitor becomes charged and the inductor loses all its stored energy. At this time the capacitor will begin to discharge though the inductor, causing the current to flow in the opposite direction. This in turn causes a new magnetic field to form around the inductor until the capacitor becomes fully discharged. The current will continue to flow as the magnetic field collapses but will diminish to zero again as the capacitor becomes charged again.

If the switch remains in position 2 then the current will continue to flow back and forth through the capacitor and the inductor in an oscillatory manner with a frequency determined by the values L and C of the inductor and capacitor. During this process the capacitor becomes charged with alternating polarity.
In practice the amplitude of these oscillations will diminish in time because of energy losses caused by resistances inherent in the circuit components. By applying Kirchhoff's current law and Kirchhoff's voltage law and neglecting inherent resistances, the current can be modelled by the second order differential equation

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{I}{C} = 0.$$

An RLC circuit

Fig. 1, on page 2 of this document, shows a circuit containing a resistor, an inductor and a capacitor together with a single-pole two-way switch and a DC voltage source. When the switch is in position 1, the behaviour of the circuit can be modelled by the second order differential equation

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{C} = \frac{\mathrm{d}V}{\mathrm{d}t}.$$

If V is constant then

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{C} = 0.$$

This equation also applies to the case when the switch is in position 2 i.e. when the source voltage is disconnected. The actual current flowing around the circuit will diminish to zero in both cases if the switch remains in the same position for a sufficient length of time.
When the position of the switch is changed suddenly there will be a flow of current for a short time as the capacitor charges or discharges, but this current will eventually diminish to zero again. The actual behaviour of the current flow, which could be oscillatory during this transient period, will depend on the component values, the instantaneous conditions at the time the switch position is changed and the time elapsed after this.

Appendix

Standard solutions of second order differential equations with constant coefficients

A general solution to the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

where a, b and c are constants, may be found as follows.

Define an auxiliary equation in the form of a quadratic in m

$$am^2 + bm + c = 0.$$

If m_1 and m_2 are the solutions to this auxiliary equation, then a solution to the differential equation is one of the following

1 when m_1 and m_2 are real and different then

$$y = A\mathrm{e}^{m_1x} + B\mathrm{e}^{m_2x}$$

where A and B are constants,

2 when m_1 and m_2 are real and the same then

$$y = \mathrm{e}^{m_1x}(A + Bx)$$

3 if m_1 and m_2 are complex in the form $p \pm \mathrm{j}q$ then

$$y = \mathrm{e}^{px}(A\cos qx + B\sin qx).$$

The constants A and B may be established in particular cases by substituting initial conditions

$$y = y_0 \text{ when } x = x_0$$

and $$\frac{\mathrm{d}y}{\mathrm{d}x} = y'_0 \text{ when } x = x_0$$

where y_0 and y'_0 are known values for $x = x_0$.