OCR

Tuesday 24 June 2014 – Morning

A2 GCE MATHEMATICS (MEI)

4777/01 Numerical Computation

Candidates answer on the Answer Booklet.

OCR supplied materials:
- 12 page Answer Booklet (OCR12) (sent with general stationery)
- MEI Examination Formulae and Tables (MF2)
- Graph paper

Other materials required:
- Scientific or graphical calculator
- Computer with appropriate software and printing facilities

Duration: 2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the Answer Booklet. Please write clearly and in capital letters.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- Additional sheets, including computer print-outs, should be fastened securely to the Answer Booklet.
- Do **not** write in the bar codes.

COMPUTING RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities throughout the examination.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.
  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the formulae in the cells as well as the values in the cells.
  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 **(i)** The equation $f(x) = 0$ has a root α. The equation is rearranged to $x = g(x)$.

State the condition for the iteration $x_{r+1} = g(x_r)$ to converge to α. (You may assume that a suitable starting value is used.)

Obtain the relaxed iteration

$$x_{r+1} = (1-\lambda)x_r + \lambda g(x_r).$$

Find, in terms of α, the best value of λ. How would a value of λ be chosen in practice? **[5]**

(ii) Show graphically that the equation

$$kx = \exp\left(x + \frac{1}{x}\right), \qquad (*)$$

where $k = 10$, has two positive roots.

Show also that (*) with $k = 10$ has no negative roots.

Show that the iteration

$$x_{r+1} = \frac{1}{10}\exp\left(x_r + \frac{1}{x_r}\right)$$

converges to one of the roots but diverges from the other.

Use the method of relaxation to find the other root correct to 4 decimal places. **[14]**

(iii) Obtain, correct to 4 decimal places, both roots of (*) in the case $k = 20$. **[5]**

2 In the table, the values of x are exact and the values of y are subject to experimental error.

x	1	2	3	4	5
y	7.24	12.15	13.84	12.25	7.07

It is thought that the relationship between y and x can be modelled as a quadratic function with $y = 0$ when $x = 0$. That is, $y = bx + cx^2$ for some constants b and c. These constants are to be estimated using least squares.

(i) Show, using partial differentiation, that one of the normal equations is

$$\Sigma xy = b\Sigma x^2 + c\Sigma x^3.$$

Obtain the other normal equation. **[6]**

(ii) Using a spreadsheet and the given data,

- find values for b and c,
- draw a graph of the data points and the fitted curve,
- find the sum of the residuals and the sum of the squares of the residuals. **[14]**

(iii) State why the sum of the residuals is not zero, and explain how this relates to the assumptions about the relationship between y and x.

Suppose the model is modified so that y is still a quadratic function of x, but without y being zero when x is zero. State, with a reason, what effect this change will have on the sum of squares of the residuals. (You are not required to do any further calculations.) **[4]**

3 **(i)** The trapezium rule, using n strips of equal width h, is used to find an estimate T_n of the integral

$$I = \int_a^b \mathrm{f}(x)\,\mathrm{d}x.$$

The global error in T_n is of the form

$$A_2h^2 + A_4h^4 + A_6h^6 + \ldots,$$

where the coefficients A_2, A_4, A_6, … are independent of n and h.

Show that $T_n{}^* = \frac{1}{3}(4T_{2n} - T_n)$ is an estimate of I with global error of order h^4.

Write down, without proof, an expression, $T_n{}^{**}$, in terms of $T_{2n}{}^*$ and $T_n{}^*$, that will be an estimate of I with global error of order h^6. **[6]**

(ii) Set up a spreadsheet that uses Romberg's method to find the value of the integral

$$I = \int_0^{\frac{\pi}{2}} \sqrt{k + \sin x}\,\mathrm{d}x$$

for $k = 1$. Your method should show the values of T, T^* and T^{**}.

Show that, when $k = 1$, $I = 2$ (to at least 7 decimal places).

Show by means of ratios of differences that T, T^* and T^{**} have errors consistent with your working in part **(i)**. **[12]**

(iii) Modify your spreadsheet to obtain the values of I for $k = 0.8$, 0.4, 0.2 and 0.

Comment on the accuracy of your estimates and the ratios of differences as k gets smaller. **[6]**

Question 4 begins on page 4.

4 In this question, the Gauss-Seidel iterative method is to be used to solve the matrix equation

$$\mathbf{Mx} = \mathbf{c}, \qquad (*)$$

$$\text{where } \mathbf{M} = \begin{pmatrix} a & 1 & 0 & -1 \\ 2 & b & 2 & 0 \\ 0 & 2 & b & 2 \\ 1 & 0 & -1 & a \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}.$$

(i) State the conditions on a and b for **M** to be diagonally dominant.

State the conditions on a and b for **M** to be strictly diagonally dominant.

Explain the relevance of these conditions for the convergence of the Gauss-Seidel method. **[6]**

(ii) For the case $a = 2$ and $b = 4$, use the Gauss-Seidel method on a spreadsheet to obtain a numerical solution for (*). Deduce the exact solution as fractions and show that your deduction is correct. **[10]**

(iii) Show, by means of examples, that larger values of a lead to faster convergence. Show similarly that larger values of b also lead to faster convergence. **[4]**

(iv) For the case $a = 2$, find correct to 1 decimal place the largest value of b for which the Gauss-Seidel method diverges. **[2]**

(v) For the case $b = a^2$, find correct to 1 decimal place the largest value of a for which the Gauss-Seidel method diverges. **[2]**

END OF QUESTION PAPER