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LEVEL 3 CERTIFICATE

Topic Exploration Pack

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QUANTITATIVE PROBLEM SOLVING (MEI)
QUANTITATIVE REASONING (MEI)

Exponential Growth and Decay

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MEI
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OCR
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This Topic Exploration Pack should accompany the OCR resource '*Exponential growth and decay'* learner activity, which you can download from the OCR website.

This activity offers an opportunity for maths skills development.

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Introduction

There are three aspects of the exponential growth and decay that need to be taught in Introduction to Quantitative Reasoning:

Firstly, learners need to explore exponential growth and decay, including interpreting output from spreadsheet. Contexts in which these ideas need to be explored include borrowing and saving money, bacterial growth and radioactive decay.

Secondly, learners need to be able to represent and interpret exponential growth and decay in a graph. Learners may be asked to plot or sketch exponential graph and are expected to be familiar with the idea of half life and its graph representation as $y = ka^x$ where both k and a are constant.

Finally, they should be able to solve equations of the forms. $x^5 = 35$ and $1.05^x = 8.2$ using trial and improvement for the last one.

Exponential growth and decay is a concept that appears in a wide number of real-life settings. In the sciences it appears whenever a natural quantity grows or decays at rate depending on how much of the quantity there is at a given moment in time. Take for example the cooling of a cup of coffee after it has boiled. You would expect the temperature of the coffee to decrease from 100°C to 22°C (room temperature). At the beginning the coffee will cool quickly but as time goes on the coffee will cool slower as the temperature decreases. The main point here is that the higher the temperature the quicker the rate the temperature changes.

This is a key aspect of exponential growth/decay and is observed in many situations:

- Bacteria growth – the more bacteria there are the quicker they grow; hence exponential growth.
- Investing money in a compound interest scheme – the more money there is, the quicker the money increases; hence exponential growth.
- Radioactive decay – the higher the mass, the quicker the mass decrease; hence exponential decay.

Mathematically this can be written as:

$$\text{Rate 'stuff' changes } \alpha \text{ Amount of 'stuff'}$$

The symbol α means ‘proportional to’. Identifying when a quantity is changing exponentially is an important skill for learners to grasp and they wouldn’t be familiar with this from GCSE.

The mathematical analysis of this concept requires some advanced ideas but a quantity *N* that changes exponentially can be written in a relationship as:

$$N = ka^x$$

where *k* is the original amount a is a fixed number and *x* is the independent variable that changes, often time. The fundamental difference to equations that learners would have seen at GCSE is that the independent variable is a *power* and not like x^2 which learners will have been familiar with at GCSE. Therefore learners will need to find strategies to be able to solve equations like the one above.

Prior Knowledge

Learners should be able to:

i) Plot a graph from a linear/quadratic relationship by evaluating the *y* coordinate at various *x* values.
ii) Solve equations which do not have an exact method by trial and improvement, for example $x^3 - 2x + 1 = 0$ by repeatedly substituting values and comparing.
iii) Be able to calculate and solve problems involving compound interest, in particular the formula:

$$N = A\left(1+\frac{x}{100}\right)^n$$

where A is the original amount, *n* is the time period, *x* is the percentage interest rate and *N* is the final amount. Note that $\left(1+\frac{x}{100}\right)$ is simply the multiplier.

Teaching points/Misconceptions

A key area of difficulty in this topic is percentage increases and decreases. This is often because learners are 'over-taught' the written method of how to find percentage increases and decreases, when the calculations are relatively easy and shouldn't provide them with too much fear. The key is to understand that the multiplier 1 represents a change of 0%. A multiplier of 1.43 therefore represents an increase of 43% whilst a multiplier of 0.83 represents a decrease of 17% (note that it is the difference between the multiplier and 1 which is the change – it isn't a percentage decrease of 83%). Amounts and percentages can then be found using a very simple formula:

$$\textit{Original Amount} \times \textit{Multiplier} = \textit{New Amount}$$

This formula can be stated in a formula triangle and then applied to situations where the percentage change is required. For example if initially the mass of a radioactive isotope is 5.6 grams and after a week it is 4.7 grams then to work out the percentage decrease we have:

$$5.6 \times \textit{Multiplier} = 4.7$$
$$\textit{Multiplier} = \frac{4.7}{5.6} = 0.84$$

This represents a decrease of (1–0.84) = 16%.

Also say we wanted to know the population of rabbits which were initially 82 and experienced an increase of 16% then:

$$82 \times \textit{Multiplier} = \textit{New Population}$$
$$82 \times 1.16 = \textit{New} = 95.12$$

Rearrangement using the formula triangle will also help learners find the original amount when the new amount and percentage change given:

$$\textit{Original Amount} = \textit{New Amount} \div \textit{Multiplier}$$

Useful Links

Exponential growth and doubling time is a very useful resource aiming to support three areas of exponential growth: Compound interest, Population growth and Bacteria growth. It includes videos, tasks using spreadsheets and assessment ideas.

Part A Borrowing and saving money

Bank loans that are based on charging compound interest end up costing the borrower much more than the original loan amount. In this situation the borrower not only pays interest on the money borrowed, but interest on the interest being charged.

Saving accounts are based on adding compound interest to the original amount invested. In this situation the saver not only earns interest on the original amount invested, but also earns interest on each subsequent amount of interest. This means that the amount of interest must be calculated on the original amount plus the previous year's interest. An amount of money that is increasing every year, month, or week by the same multiple is called exponential growth.

Exponential growth or decay can be expressed as an equation of a form :

$$N = A\left(1+\frac{x}{100}\right)^n$$

where A is the initial value, N is the new amount after the total interest has been added, x is the percentage interest and n is the number of time periods the money is borrowed for or invested in. This means that if there is an annual 17.5% interest rate for a credit card the equation becomes:

$$N = A \times 1.175^n$$

Activity A

You are investing some money (£100) into a compound interest account but you don't know which one to choose as there are a number of different banks offering different deals:

Bank 1: Interest rate 50% over each 1/2 year

Bank 2: Interest rate 25% over each 1/4 year

Bank 3: Interest rate 12.5% over each 1/8 year

Bank 4: Interest rate 6.25% over each 1/16 year

After 1 year which one provides the most interest on your original investment of £100?

What happens if you continue the pattern of halving the interest rate and the doubling the number of times you receive interest?

Notes

Learners should initially multiply the previous amount by the multiplier to get the new amount. You might want to go over a generic example first; 17.5% APR on an investment of £300 with each payment occurring over a month. The key thing here is that once learners have done a couple of examples they should be able to very quickly calculate the final amounts for the last one by recreating the compound growth formula for themselves, whether formally or actually. Asking learners to write down and explain what they are doing will demonstrate their understanding on the subject. It should be noted that this is not how banks do business!!! They actually divide the annual percentage rate by 12 and calculate at the end of each month; this should be explained to the learners.

1) Complete the tables for Banks 1 and 2:

BANK 1 – 2 period 50% each	
Time Period	Amount
1	100 × 1.5 = 150
2	100 × 1.5 = 225

BANK 2 – 4 period 25% each	
Time Period	Amount
1	100 × 1.25 = 125
2	125 ×1.25 = 156.25
3	156.25 × 1.25 = 195.3125
4	195.3125 × 1.25 = 244.140625

2) Find the amounts at the end for banks 3 and 4 by drawing a table or a direct calculation.

Bank 3: $100 \times (1.125)^8 = 256.578$

Bank 4: $100 \times (1.0625)^{16} = 263.793$

3) What happens if we continue the process of halving the interest and doubling the times it is paid? Continue the process three more times

If we halve the interest and double the time periods the end amount is $100 \times 1.03125^{32} = 267.70$

If we do this again it is $100 \times 1.015625^{64} = 269.73$

4) Create a spreadsheet that will calculate what will happen if this process is continued say 100 times. What happens?

When learners continue the pattern of halving the interest rate but doubling the time periods then they get the following sequence of numbers:

Interest Rate	Time periods	Final Amount
0.0625	16	263.7928497
0.03125	32	267.6990129
0.015625	64	269.7344953
0.0078125	128	270.773902
0.00390625	256	271.2991624
0.001953125	512	271.5632
0.000976563	1024	271.6955729

5) Divide the final amount by 100 and type this number into an internet search engine. What do you find? Research this special number.

The final amount will actually converge to $100 \times e$ where e is exponential e named by the mathematician Leonard Euler which has a value of 2.7182818… This could be explored as a research task at the end for early finishers.

Part B – Cooling

Exponential decay can occur in many naturally occurring situations. In this activity learners do an experiment to see if the exponential model is a correct one.

Activity B

This activity relies on how adventurous the teacher wants to be in choosing the material to measure the cooling off. One way is to use a freshly boiled cup of coffee or a more interesting way would be order a pizza for the beginning of the lesson and measure the rate at which it cools. Of course it depends on your budget!

Equipment:

- Thermometer
- Stopwatch
- Something to cool – cup of coffee/pizza etc.

At the beginning of the lesson explain to learners that the aim is to predict when the coffee/pizza will cool to room temperature. Explain that it should cool exponentially; in a set moment of time the temperature will decrease by a constant amount.

Take the temperature in front of the whole class and ask one learner to start the stopwatch. After 1 minute take the temperature again. This will give you the decrease in temperature after one minute. Ask one learner to continue taking the temperature after every minute and recording it in a table like the one below.

Now explain to the class how they will calculate the time for the final temperature. It is explained with an example. Newton's law of cooling states that the temperature decreases at a rate proportional to the difference of the temperature from the room temperature. This means that the numbers calculated have to be the difference between the temperature of the coffee/pizza and the room temperature (22 degrees). So for example although the temperature is initially at 100 degrees, the value of importance is the difference between the room temperature, ie.100 – 22 = 78.

At t = 0 the temperature was 100°C, after 1 minute the temperature was 96.2°C. Therefore in one minute of time we have to work out the decrease in temperature as a percentage. You could ask learners to calculate this for themselves or explain with an example and then get them to do it for the real example.

Difference with room temp = 100 – 22 = 78

Difference with room temp = 96.2 – 22 = 74.2

Change = 78 – 74.2 = 3.8

$$\% \text{ difference} = \frac{3.8}{78} = 0.05$$

Multiplier = 1 – 0.05 = 0.95

Therefore learners can fill in the following table:

Time (minutes)	Difference in Temp between object and the room
0	78
1	74.2
2	74.2 × 0.95 = 70.32
4	
5	
6	
7	
8	
9	
10	

This is achieved by multiplying the previous amount by the multiplier 0.95. What we are interested in however is the time it takes for the temperature to reach room temperature 22°C. This can be achieved by solving the equation:

$$T = 78 \times 0.95^n$$

Where n is the number of minutes. This is done by a ‘trial and improvement’ method. Learners first make a guess, say 20 minutes, they substitute it into the equation to get:

$$T = 78 \times 0.95^{20} = 27.96$$

This isn’t small enough so try $n = 35$:

$$T = 78 \times 0.95^{35} = 12.95$$

Learners should now go into decimals and try and get their answer as close to 0°C as possible.

Once this has been done ask learners to create a graph of the decrease in the difference of the temperature using a spreadsheet or on paper. Make sure they draw the curve smoothly between points. Learners can use their table and the points they calculated using the trial and improvement approach.

Once this has been achieved the class can see if this time is indeed correct by comparing their temperatures to the actual temperatures. Display the results on the board and ask learners to create another graph on the same axis as their theoretical results.

How closely do they match up?

How did their theoretical temperatures at the end of the lesson match up with the actual temperatures?

Part C Population Growth

Activity C

This activity brings ideas of how populations of organisms can grow according to an exponential rule. There are many examples of how exponential growth can occur in populations; bacteria growth is the obvious example. In this activity we look at rabbit populations and different models used to estimate the population growth.

There are initially 10 rabbits in a field. A number of models are proposed to model their growth which learners are going to investigate.

Model 1:

The rabbit population grows exponentially at a fixed rate of multiplying by a factor of 1.46 each year.

1) Find the population at the end of each year for the next 10 years

10,15,21,31,45,66,97,141,207,301

2) Write down the equation for the number of rabbits R after n years

$R = 10 \times 146^{n}$

3) How many years does it take for there to be a million rabbits in the field?

n = 30.42 years

4) What is wrong with this model?

The model predicts that the rabbits increase with an ever quicker rate. The population will continue to get larger and larger forever. This is an unrealistic situation as the population will be bounded by the size of the field and available resources etc.

Model 2:

The rabbit population grows exponentially according to the rule:

$$R = 400 - 390 \times 1.01^n$$

1) Find the population after 10 years, 20 years, 30 years, 40 years and 50 years. using a calculator and substituting the values in.

47,80,111,138,163

2) How many years does it take for there to be a million rabbits in the fields?

The rabbit population will never reach a million. Learners should see that the higher the value of n that they substitute into the equation the closer the rabbit population reaches 400.

3) What happens in this model? By plotting the graph of R against n in multiples of 10 what can you say happens to the population in the long term?

See graph attached. The graph clearly shows that as the number of years increase the number of rabbits approaches 400 but never surpasses it. Note as $n \to \infty$, $R \to 400$; it will never actually reach 400, 400 represents the 'capacity' of the field based on the size of the field and the availability of resources.

Part D Radioactive Decay

Activity D

The half-life of a radioactive substance is the time taken for a given amount of the substance to become reduced by half because of radioactive decay.
There are two definitions of half-life, but they mean the same in practice.

Half-life is the time taken for:

- The number of atoms in a sample to halve because of radioactive decay
- The Geiger counter count rate from a sample to fall to half its starting level.

The half live of the radioactive element Caesium -134 is 2 hours, so that starting with a Geiger counter reading of 64 clicks a second for a sample of Caesium -134.

- 2 hours later it will be 32 clicks per second
- 4 hours after the start it will be 16 clicks per second
- 8 hours after the start it will be 8 clicks per second
- And so on …

Use the information above to calculate how many clicks for these times after starting.

(a) 32 hours
(b) *n* hours
(c) 3 hours.

Experience has shown that using and applying the concept of a half-life including using it to form expressions can, although superficially understood, present a real challenge.

It is advisable to start this activity by making a table to show how the amount of clicks is being halved every two hours; this will also help to find the amount of clicks after 32 hours. Part (b) is an essential step to understand and solve part (c). Again explanation needs to be provided to why the power is $\frac{n}{2}$. This is because for every *two hours* we multiply by 0.5. For 8 hours we multiply 64×0.5^4. The time period in this case is every two hours and hence this has to be taken account in the formula. A point of discussion may be why the answer for part (c) is not 24?

Answers:

(a) 2 (clicks per second)
(b) $64 \times 0.5^{\frac{n}{2}}$ clicks per second where n is in hours
(c) 22.6 … (clicks per second).

Part E Pollution

Exponential growth is the growth in which some quantity, such as population size or economic output, increases at a constant rate per unit of time. The connection between these two is that exponential growth plays a key role in five important and interconnected environmental problems and issues by having population growth, resource use and waste, poverty, loss of biological diversity, and global climate change.

Activity E

Pollution

There is a spillage of a chemical into a lake.

The average concentration of the chemical in the lake is now 10 times the recommended level.

The concentration of chemical will be expected to decrease gradually due to natural changes, such as fresh water entering the lake (as rainfall) and water/chemical mixture flowing out of the lake (in rivers). If it is predicted that the chemical concentration will decrease by 25% every month from its concentration at the beginning of that month.

How long will it take for the lake to return to the recommended level of concentration?

There are two ways of solving this simple problem:

Learners can reduce the average concentration by 25% each month (multiplier 0.75):

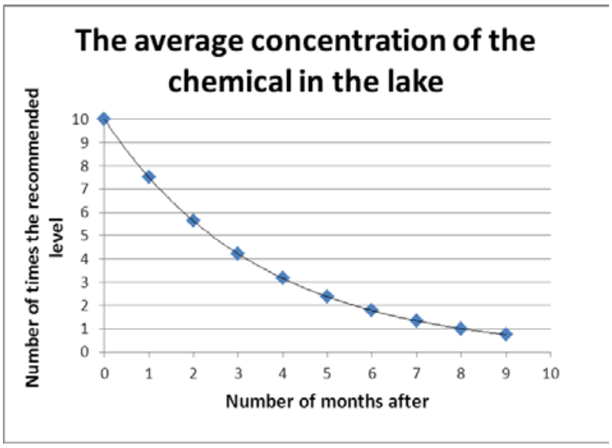
Now	10 times the level
After 1 month	7.5 times the level
After 2 months	5.625 times the level
After 3 months	4.219 times the level
After 4 months	3.164 times the level
After 5 months	2.373 times the level
After 6 months	1.780 times the level
After 7 months	1.335 times the level
After 8 months	1.001 times the level
After 9 months	0.751 times the level

Learners can try solving an equation $1 = 10 \times (0.75)n$ using trial and improvement:

$n = 5$	$10 \times (0.75)^5 = 2.373$
$n = 7$	$10 \times (0.75)^7 = 1.335$
$n = 8$	$10 \times (0.75)^8 = 1.001$
$n = 9$	$10 \times (0.75)^9 = 0.751$

It will take 8 months for the lake to return to the recommended level of concentration.

Learners should be encouraged to draw an exponential graph to represent their findings.



Cloudy water

Pollution makes the water in the lake cloudy.

This is an important consideration for scuba-divers, who always dive in pairs and must maintain visual contact for safety reasons.

The intensity of light is reduced by 10% for each 20 cm of water it travels through.

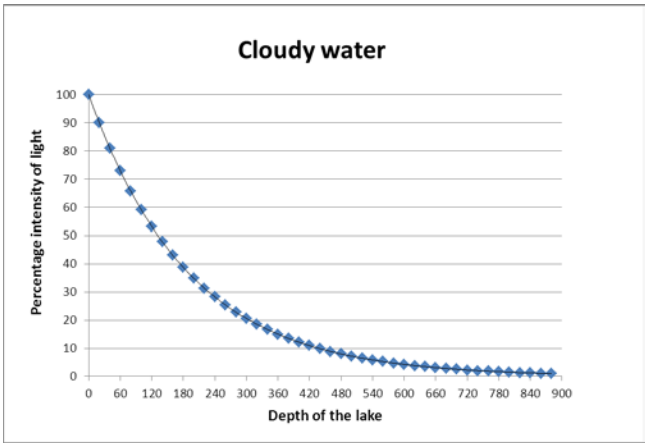
At what depth in the lake will the intensity of light be just 1% of that on the surface?

Again, this can be solved in many different ways including using a spreadsheet. Learners should be encouraged to find the formula of percentage decrease for each 20 cm themselves regardless of the method chosen.

The solution for this problem could be obtained from $0.01 = 0.9^{\frac{x}{20}}$ using trial and improvement method giving a depth between 860 and 880 cm (the actual answer is 870 cm –why?). Note that explanation regarding using $\frac{x}{20}$ rather than just x is needed.

Learners need to understand that the first decrease happens when the light reaches 20 cm.

Again encourage learners to draw an exponential graph to represent the function (that can be done using spreadsheet).



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