# Teacher Delivery Guide Core Pure: Complex Numbers

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Specification** | **Ref.** | **Learning outcomes** | | **Notes** | | **Notation** | **Exclusions** |
| **Y420 CORE PURE: COMPLEX NUMBERS**  **Y410 CORE PURE: COMPLEX NUMBERS** | | | | | | | |
| Language of complex numbers | Pj1 | Understand the language of complex numbers. | Real part, imaginary part, complex conjugate, modulus, argument, real axis, imaginary axis. | |  | |  |
| Complex numbers and polynomial equations with real coefficients | j2 | Be able to solve any quadratic equation with real coefficients. |  | |  | |  |
| j3 | Know that the complex roots of polynomial equations with real coefficients occur in conjugate pairs. Be able to solve cubic or quartic equations with real coefficients. | Use of the factor theorem once a real root has been determined.  Sufficient information will be given to deduce at least one complex root or quadratic factor for quartics. | |  | | Equations with degree > 4. |

***DISCLAIMER***

This resource was designed using the most up to date information from the specification at the time it was published. Specifications are updated over time, which means there may be contradictions between the resource and the specification, therefore please use the information on the latest specification at all times.If you do notice a discrepancy please contact us on the following email address: [resources.feedback@ocr.org.uk](mailto:resources.feedback@ocr.org.uk)

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Specification** | **Ref.** | **Learning outcomes** | | **Notes** | | **Notation** | | **Exclusions** |
| **CORE PURE: COMPLEX NUMBERS (a)** | | | | | | | | |
| Arithmetic of complex numbers | j4 | Be able to add, subtract, multiply and divide complex numbers given in the form , and real. | Division using complex conjugates. | |  | |  | |
| j5 | Understand that a complex number is zero if and only if both the real and imaginary parts are zero. |  | |  | |  | |
| Modulus-argument form | j6 | Be able to use radians in the context of complex numbers. | Use exact values of trigonometric functions for multiples of  and . | |  | |  | |
| j7 | Be able to represent a complex number in modulus-argument form. Be able to convert between the forms  and  where *r* is the modulus and *θ* is the argument of the complex number. |  | | is the modulus of *z*.  arg *z* for principal argument, where .  Radian measure. | |  | |
| j8 | Be able to multiply and divide complex numbers in modulus-argument form. | The identities for  and may be assumed in the derivation of these results. | |  | |  | |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Specification** | **Ref.** | **Learning outcomes** | **Notes** | | **Notation** | **Exclusions** |
| **CORE PURE: COMPLEX NUMBERS (a)** | | | | | | |
| The Argand diagram | j9 | Be able to represent and interpret complex numbers and their conjugates on an Argand diagram. |  | |  |  |
| j10 | Be able to represent the sum, difference, product and quotient of two complex numbers on an Argand diagram. |  | |  |  |
| j11 | Be able to represent and interpret sets of complex numbers as loci on an Argand diagram. | Circles of the form .  Half lines of the form .  Lines of the form .  Regions defined by inequalities based on the above e.g. . Intersections and unions of these. | | For regions defined by inequalities learners must state clearly which regions are included and whether the boundaries are included. No particular shading convention is expected. | for . |
| **CORE PURE: COMPLEX NUMBERS (b)** | | | | | | |
| De Moivre's theorem and applications | Pj12 | Understand and use de Moivre's theorem. | |  |  |  |
| j13 | Be able to apply de Moivre's theorem to finding multiple angle formulae and to summing suitable series. | | e.g. the expression of  as a rational function of .e.g. finding . |  |  |
| The form | j14 | Understand the definition  and hence the form . | |  |  |  |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Specification** | **Ref.** | **Learning outcomes** | **Notes** | **Notation** | **Exclusions** |
| **CORE PURE: COMPLEX NUMBERS (b)** | | | | | |
| The  *nth* roots of a complex number | j15 | Know that every non-zero complex number has *n* distinct *n*th roots, and that on an Argand diagram these are the vertices of a regular *n*-gon. |  |  |  |
| j16 | Know that the distinct *n*th roots of  are:  for . |  |  |  |
| j17 | Be able to explain why the sum of all the *n*th roots is zero. |  |  |  |
| Applications of complex numbers in geometry | j18 | Understand the effect of multiplication by a complex number on an Argand diagram. | Multiplication by corresponds to enlargement with scale factor *r* with rotation through *θ* about the origin. e.g. multiplication by i corresponds to a rotation of  about the origin. |  |  |
| j19 | Be able to represent complex roots of unity on an Argand diagram. | ‘Unity’ means 1. |  |  |
| j20 | Be able to apply complex numbers to geometrical problems. | e.g. relating to the geometry of regular polygons. |  |  |

# Thinking Conceptually

### General approaches

This topic breaks into a couple of sections courtesy of the division between AS and A2 Level. The opening section needs a basic explanation on the origin of complex numbers as the solution to the equation . This leads to an introduction to  and its value. This principle can then be extended to the solution of quadratic equations whose discriminant is less than zero. Once the need for complex numbers has been established then the basics of arithmetic comes next. A quick recap on the difference of two squares helps establish the use of the complex conjugate, specifically in division. It is perhaps worth noting here that there are a good number of calculators that will now allow the use of complex numbers. These can be useful to check that the right answer has been achieved. The use of an Argand diagram, often best explained as a third axis not simply replacing the usual  and  values with Re and Im, and modulus and argument values follows here. Some students may not have met radians at this stage, because they are not part of AS mathematics, though they are included in the AS Further Pure Core, so that may need introducing first. The use of  can then be derived and this helps with the section on loci. Although it is not part of the AS syllabus, a brief introduction to polar coordinates helps explain the use of the modulus and the argument of complex numbers.

If this topic has been introduced through solving quadratics then looking at the roots of equations and the square root of complex numbers becomes a more obvious progression, particularly the process of starting with the root and working back to the quadratic. A recap of the solution of simultaneous equations is a worthwhile exercise at this point.

Finally, this section concludes with the work on loci. This is very limited. In reality, there are only three options, a circle, a half line or the perpendicular bisector. If  has been carefully explained earlier then the jump to the circle formula is easier.

The second half is the stage 2 syllabus, which falls into two sections.

Firstly, the use of De Moivre’s theorem, particularly in the solution of equations. Finding the roots of unity falls into this section and is a nice way of backing up other work. The equation can then be shown to have three solutions rather than only one.

The last section involves the link to trigonometric formulae. Careful coordination with the Maths A Level course is needed as launching into this before covering the trigonometric identities can lead to significant confusion.

### Common misconceptions or difficulties learners may have

Fear tends to be the underlying problem here. The name suggests that this should be a complicated topic. Beginning the Further Mathematics course with complex numbers is a good method of demonstrating that there is nothing to be afraid of.

Care should be taken when teaching the modulus function to ensure that learners do not try to include  in the calculation as this can result in a negative value inside the square root. This can easily be explained by the use of the diagram. Similarly, the argument being measured from the positive real axis with anticlockwise being positive can lead to some confusion if this is not treated diagrammatically. The loci section should be reinforced by the transformation of graphs work making the connection between  and  .

Learners tend to have fewer problems with the stage 2 aspect of this course, although if the solution of trigonometric equations has not been thoroughly understood then difficulties will swiftly arise. A revision of this work before beginning will help avoid this issue.

### Conceptual links to other areas of the specification

Mathematics

Algebra and Functions – specifically here the solution of simultaneous equations in finding the square root of a complex number and the quadratic function. Also inequalities and curve sketching are key to the work on loci. Alongside these transformations of graphs and circles are also invaluable.

Sequences and Series – particularly the binomial expansion – is needed for the work on trigonometric identities linked to De Moivre’s theorem.

Trigonometric Functions. Much of the work on trigonometric functions is used throughout the work on complex numbers. Initially this covers the argument and the plotting of half lines. However, in the second half there is the use of trigonometric solutions to find the roots of unity and of other complex numbers and also in the work on trigonometric identities linked to de Moivre’s theorem.

Exponential and Logarithmic Functions – the work using only makes sense if this area has been properly explained.

Further Mathematics

Proof. De Moivre’s theorem can be proved by induction so a link should be made here.

Hyperbolic Functions which can be derived from trigonometric functions using complex numbers.

Further calculus – specifically the Maclaurin series which explains the links to these functions.

Polar coordinates. Although this is not a direct link it does help explain the use of modulus and argument values alongside writing  as the polar form of a complex number.

Differential Equations – the solution of a second order differential equation can be complex and this leads the work on simple harmonic motion, more specifically to damped and forced motion, which provides a nice link to the use of complex numbers.

# Thinking Contextually

The work on complex numbers is often assumed to be largely theoretical as it deals with so called imaginary numbers. However, the applications to this are considerable. Firstly, linked directly to the syllabus is the work on second order differential equations and their connection to simple, damped and forced harmonic motion.

This leads to work on earthquake damping, suspension systems on cars, sound wave dissipation and tidal wave energy. Some learners find this quite difficult to comprehend in that there is a real application to imaginary numbers. However, if the use of to create the cosine curve is demonstrated then it becomes far more apparent that this is possible.

It is possibly worth mentioning the Riemann Hypothesis and its link to prime numbers although avoid trying to go into any real detail and similarly with William Hamilton’s work on quaternions.

# Resources

| **Title** | **Organisation** | **Description** | **Ref** |
| --- | --- | --- | --- |
| [Complex Numbers](http://www.cimt.org.uk/projects/mepres/alevel/fpure_ch3.pdf) | CIMT | This is an entire chapter devoted to complex numbers. | j1-j20 |
| [Prime numbers the complex function energy levels and Riemann.](http://wwwf.imperial.ac.uk/~hjjens/Riemann_talk.pdf) | Imperial College | How complex numbers and prime numbers link in one of Maths greatest challenges. | j1-j20 |
| [Complex Numbers](https://www.mathsisfun.com/numbers/complex-numbers.html) | Maths is fun | This provides a clear and colourful introduction to complex numbers and their arithmetic.  Includes 10 standard questions and two challenging questions at the end. | j1 |
| [Complex Numbers – Introduction](http://www.purplemath.com/modules/complex.htm) | Purplemath | An introduction to complex numbers – non-colourful but very clear. | j1 |
| [Complex Numbers – Basic Definitions](https://www.intmath.com/complex-numbers/1-basic-definitions.php) | Interactive Maths | The origin and language of complex numbers  Questions with revealable answers. | j1 |
| [What is a complex number](http://www.mathcentre.ac.uk/resources/Engineering%20maths%20first%20aid%20kit/latexsource%20and%20diagrams/7_1.pdf) | Maths Centre | A nice introduction to the language of complex numbers. | j1 |
| [Complex Plane](https://www.mathsisfun.com/algebra/complex-plane.html) | Maths is fun | This looks at the Argand diagram and the idea of the complex number as a vector. It then shows how to convert the forms.  It includes 10 questions at the end. | j1 |
| [Complex Numbers Arithmetic](http://www.personal.soton.ac.uk/jav/soton/HELM/workbooks/workbook_10/10_1_cmplx_arith.pdf) | Helm | This takes the complex number onto the Argand diagram and transfers it through vectors into polar form.  It is also a workbook – with answers. | j1 |
| [Complex Numbers and the Quadratic Formula](http://www.purplemath.com/modules/complex2.htm) | Purplemath | This links complex numbers back to the solution of quadratic equations. | j2 |
| [Euler’s Formula Explained](https://betterexplained.com/articles/intuitive-understanding-of-eulers-formula/) | Better Explained | A very comprehensive if slightly extensive but well-illustrated explanation of Euler's formula. | j3 |
| [Operations on Complex Numbers](http://www.purplemath.com/modules/complex2.htm) | Purplemath | This takes the learner through arithmetic with complex numbers, step by step. | j4 |
| [Complex Numbers](http://mathworld.wolfram.com/ComplexNumber.html) | Wolfram | An introduction to complex numbers and their arithmetic. | j4 |
| [The exponential form of a complex number](http://www.personal.soton.ac.uk/jav/soton/HELM/workbooks/workbook_10/10_3_exp_fm_cmplx_nmbr.pdf) | Helm | This focuses on the basic of complex numbers arithmetic  It is also a workbook – with answers. | j4 |
| [Basic Operations with complex numbers](https://www.intmath.com/complex-numbers/2-basic-operations.php) | Interactive Maths | Arithmetic with complex numbers  Questions with revealable answers. | j4 |
| [Polar Form of a complex number](https://www.intmath.com/complex-numbers/4-polar-form.php) | Interactive Maths | The conversion between polar and Cartesian form  Questions with revealable answers. | j9 |
| [Graphical Representation of Complex Numbers](https://www.intmath.com/complex-numbers/3-graphical-representation.php) | Interactive Maths | Plotting complex numbers on an Argand diagram  Questions with revealable answers. | j9, j10 and j11 |
| [The Mandelbrot Set](http://www.fractal-explorer.com/mandelbrotset.html) | Fractal Explorer | A nice and simple explanation of how complex numbers and fractals combine. | j11 |
| [Mandelbrot Unveiled](https://plus.maths.org/content/unveiling-mandelbrot-set) | Plus Maths | A comprehensive explanation of the mandelbrot and Julia sets origins in complex numbers. | j11 |
| [Complex Numbers Loci](http://furthermaths.org.uk/files/sample/files/ComplexLoci.html) | FMSP | An interactive exercise on complex numbers and loci  Very clear and very visual. | j11 |
| [Locus of Points from modulus of complex numbers](https://www.geogebra.org/m/Xch9MRTM) | Geogebra | This is an interactive applet that illustrates the link between the modulus function of complex numbers and loci. | j11 |
| [Powers and roots of complex numbers](https://www.intmath.com/complex-numbers/7-powers-roots-demoivre.php) | Interactive Maths | De Moivre's Theorem and its applications  Questions with revealable answers. | j12 and j13 |
| [Complex Numbers](http://www.personal.soton.ac.uk/jav/soton/HELM/workbooks/workbook_10/10_1_cmplx_arith.pdf) | Helm | This tackles conversion between the cartesian and exponential form alongside an explanation of its origin in the Maclaurin power series.  It also links to hyperbolic trigonometric functions at the end.  It is also a workbook – with answers. | j14 |
| [Products and Quotients of complex numbers](https://www.intmath.com/complex-numbers/6-products-quotients.php) | Interactive Maths | Arithmetic with complex numbers in exponential form  Questions with revealable answers. | j14 |
| [Multiplication and division in polar form.](http://www.mathcentre.ac.uk/resources/Engineering%20maths%20first%20aid%20kit/latexsource%20and%20diagrams/7_6.pdf) | Maths Centre | Exactly what it says on the tin  This is an engineering first aid kit and so shows that complex numbers extend beyond the purely imaginary. | j14 |
| [Introducing Quaternions](http://math.ucr.edu/~huerta/introquaternions.pdf) | Fullerton College | A link between complex numbers in their polar form and William Hamilton’s Quaternions. | j14 |
| [Argand diagrams and the polar form](http://www.personal.soton.ac.uk/jav/soton/HELM/workbooks/workbook_10/10_2_argnd_dgm_polar_fm.pdf) | Helm | De Moivre's Theorem and its application.  It is also a workbook – with answers. | j14 |
| [Nth roots](http://www.wolframalpha.com/widgets/view.jsp?id=bda7069a6ac3481b27c9986c9bc51e49) | Wolframalpha | An applet that finds and plots the nth root of any number. | j15, j16 and j17 |
| [Complex number n-th root calculator](http://www.mathforyou.net/en/online/numbers/complex/root/) | Maths for You | A step by step nth root calculator with very extensive working. | j15, j16 and j17 |
| [Nth Roots of a complex number](https://www.geogebra.org/m/d2cGh4am) | Geogebra | This applet plots the nth roots for any number.  Very useful for showing the geometrical interpretation. | j15, j16 and j17 |

**OCR Resources**: *the small print*OCR’s resources are provided to support the delivery of OCR qualifications, but in no way constitute an endorsed teaching method that is required by the Board, and the decision to use them lies with the individual teacher. Whilst every effort is made to ensure the accuracy of the content, OCR cannot be held responsible for any errors or omissions within these resources.

© OCR 2018 - This resource may be freely copied and distributed, as long as the OCR logo and this message remain intact and OCR is acknowledged as the originator of this work.

OCR acknowledges the use of the following content: n/a

Please get in touch if you want to discuss the accessibility of resources we offer to support delivery of our qualifications: [resources.feedback@ocr.org.uk](mailto:resources.feedback@ocr.org.uk)

We’d like to know your view on the resources we produce. By clicking on [‘Like’](mailto:resources.feedback@ocr.org.uk?subject=I%20liked%20the%20AS%20and%20A%20Level%20Further%20Mathematics%20B%20(MEI)%20Teacher%20Delivery%20Guide%20Core%20Pure%20Complex%20Numbers) or ‘[Dislike’](mailto:resources.feedback@ocr.org.uk?subject=I%20disliked%20the%20AS%20and%20A%20Level%20Further%20Mathematics%20B%20(MEI)%20Teacher%20Delivery%20Guide%20Core%20Pure%20Complex%20Numbers) you can help us to ensure that our resources work for you. When the email template pops up please add additional comments if you wish and then just click ‘Send’. Thank you.

Whether you already offer OCR qualifications, are new to OCR, or are considering switching from your current provider/awarding organisation, you can request more information by completing the Expression of Interest form which can be found here: [www.ocr.org.uk/expression-of-interest](http://www.ocr.org.uk/expression-of-interest)

Looking for a resource? There is now a quick and easy search tool to help find free resources for your qualification:   
[www.ocr.org.uk/i-want-to/find-resources/](http://www.ocr.org.uk/i-want-to/find-resources/)