OCR
Oxford Cambridge and RSA

GCE

Mathematics

Unit **4726**: Further Pure Mathematics 2

Advanced GCE

Mark Scheme for June 2017

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This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners’ meeting before marking commenced.

All examiners are instructed that alternative correct answers and unexpected approaches in candidates’ scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the report on the examination.

OCR will not enter into any discussion or correspondence in connection with this mark scheme.

Annotations and abbreviations

Annotation in scoris	**Meaning**
✓and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Other abbreviations in mark scheme	**Meaning**
DM1	Method mark dependent on a previous mark
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working

a Annotations should be used whenever appropriate during your marking.

The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks. It is vital that you annotate standardisation scripts fully to show how the marks have been awarded.

For subsequent marking you must make it clear how you have arrived at the mark you have awarded

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct *solutions* leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly.

Correct but unfamiliar or unexpected methods are often signalled by a correct result following an *apparently* incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, award marks according to the spirit of the basic scheme; if you are in any doubt whatsoever (especially if several marks or candidates are involved) you should contact your Team Leader.

c The following types of marks are available.

M
A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, eg by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

A
Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B
Mark for a correct result or statement independent of Method marks.

Unless otherwise indicated, marks once gained cannot subsequently be lost, eg wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

d When a part of a question has two or more 'method' steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation 'dep *' is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.

e The abbreviation ft implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only — differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, exactly what is acceptable will be detailed in the mark scheme rationale. If this is not the case please consult your Team Leader.

Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be 'follow through'. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

f Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise. Candidates are expected to give numerical answers to an appropriate degree of accuracy, with 3 significant figures often being the norm. Small variations in the degree of accuracy to which an answer is given (e.g. 2 or 4 significant figures where 3 is expected) should not normally be penalised, while answers which are grossly over- or under-specified should normally result in the loss of a mark. The situation regarding any particular cases where the accuracy of the answer may be a marking issue should be detailed in the mark scheme rationale. If in doubt, contact your Team Leader.

g **Rules for replaced work**

If a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests.

If there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others.

NB Follow these maths-specific instructions rather than those in the assessor handbook.

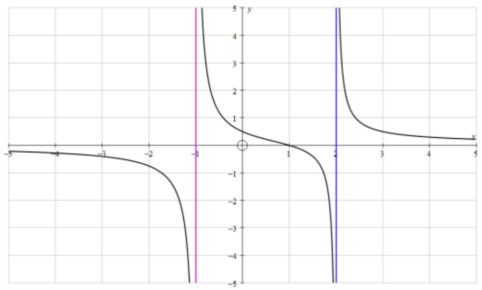
h For a *genuine* misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate's data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question.

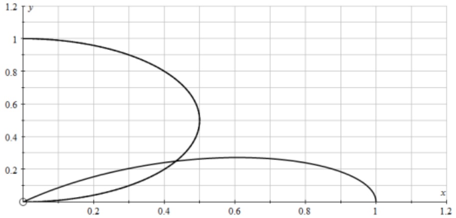
Note that a miscopy of the candidate's own working is not a misread but an accuracy error.

Question			Answer	Marks	Guidance	
1			$y = 3\cosh x - 2\sinh x \Rightarrow \frac{dy}{dx} = 3\sinh x - 2\cosh x$	M1	Correct diffn and **soi** set = 0	Alternatively: $y = \frac{1}{2}\left(e^x + 5e^{-x}\right)$
			$= 0 \Rightarrow \tanh x = \frac{2}{3}$	A1		$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(e^x - 5e^{-x}\right) = 0$ when $e^x = 5e^{-x}$
			$\Rightarrow x = \frac{1}{2}\ln\left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right) = \frac{1}{2}\ln 5$	A1	$= \ln\sqrt{5}$ www	$\Rightarrow e^{2x} = 5 \Rightarrow x = \frac{1}{2}\ln 5$ If y is wrong then M0
				[3]		

Question			Answer	Marks	Guidance	
2			$I = \int_3^4 \frac{1}{x^2 - 6x + 10}\,dx = \int_3^4 \frac{1}{(x-3)^2 + 1}\,dx$	M1 A1	Complete the square. Denominator	Sight of $(x\pm3)^2 + 1$
			From formula book: $I = \left[\tan^{-1}(x-3)\right]_3^4 = \tan^{-1}1 \pm \tan^{-1}0$	M1	Use standard form and apply limits	Accept $\tan^{-1}(x\pm3)$
			$= \frac{\pi}{4}$	A1	**www**	SC correct answer only B1
				[4]		

Question			Answer	Marks	Guidance	
3	(i)		$y=\tan^{-1}\left(\frac{x}{x+1}\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+\left(\frac{x}{x+1}\right)^2}\cdot\frac{(x+1).1-x.1}{(x+1)^2}$	M1	Attempt to apply chain rule and deal with quotient	$\tan y=\frac{x}{x+1} \Rightarrow \sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{(x+1)^2}$ M1 for implicit diffn A1 for expression for y'
			$=\frac{1}{(x+1)^2+x^2}=\frac{1}{2x^2+2x+1}$	A1	Soi	$\Rightarrow \sec^2 y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\sec^2 y\tan y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-\frac{2}{(x+1)^3}$ A1
			$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{4x+2}{\left(2x^2+2x+1\right)^2}$	A1		Substitute $x=0,\ \tan y=0, \sec y=1$
			$(\text{When } x=0,)\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(-\frac{2}{(1)^2}\right)=-2$	A1	www AG	$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2$ A1
				[4]		
	(ii)		$y_0=0,\ y_0{}'=1$ www in part (i) soi	DB1	Use values from (i)	Dependent on correct working in (i)
			$\Rightarrow (y=)\,x-x^2$	B1ft		i.e. $y = 0 + kx - x^2$ where k is *their* y'
				[2]		

Question			Answer	Marks	Guidance	
4	(i)		$x = 2$, $x = -1$, $y = 0$	B3	B1 for each	
				[3]		
	(ii)		(1, 0)	B1	Allow $x = 1$, $y = 0$ but not just $x = 1$	No extras
				[1]		
	(iii)		$\frac{dy}{dx} = \frac{(x^2-x-2)1-(x-1)(2x-1)}{(x^2-x-2)^2}$ $=0$ when $x^2-x-2-2x^2+3x-1=0$ $\Rightarrow x^2-2x+3=0$ $"b^2-4ac" = 4-12 = -8 < 0$ so no (real) roots (so gradient fn $\neq 0$) Alternative $y = \frac{1}{3(x-2)} + \frac{2}{3(x+1)}$ $\Rightarrow \frac{dy}{dx} = -\frac{1}{3(x-2)^2} - \frac{2}{3(x+1)^2} < 0$ for all values of x	M1 A1 A1 A1	Diffn dealing with quotient and set = 0 Or $(x-1)^2 = -2$ M1 A1 A1 A1www	As a quadratic in x: $x^2y - x(y+1) + (1-2y) = 0$ Coincident roots if $(y+1)^2 = 4y(1-2y)$ $\Rightarrow 9y^2 - 2y + 1 = 0$ This has no real roots So there is no value of y which gives coincident values of x. so no turning points. M1 A1 for writing as quadratic in x and setting up coincident roots $"b^2-4ac(=0)"$ A1 quadratic equation in y A1 no roots
				[4]		
	(iv)			B1 DB1	General shape, in three parts, gradient always negative Dep on 1st B Asymptotes shown and intercepts evident	If rotated then ok providing axes are labelled. i.e. x axis, halfway between 0 and 2 and y axis either a clear scale or point marked
				[2]		

Question			Answer	Marks	Guidance	
5	(i)			**B1** **B1** **B1** **DB1**	first curve approximately correct second curve approx correct intersections on "axes" curves intersect approximately in the right place, dep on 1st two B marks	Ignore any extra parts of curves
				[4]		
	(ii)		$r = 0.5$ $\theta = \dfrac{\pi}{6}$	**B1** **B1**	 Accept $\left(0.5, \dfrac{\pi}{6}\right)$ or $\left(\dfrac{\pi}{6}, 0.5\right)$	
				[2]		

Question			Answer	Marks	Guidance	
	(iii)		Limit of $\frac{\pi}{6}$	B1	Seen anywhere in either integral	$\frac{1}{2}\int\left(r_1^2 - r_2^2\right)\mathrm{d}\theta$ could earn the first 6 marks
			For first: $A=\frac{1}{2}\int_0^{\pi/6} r^2\,\mathrm{d}\theta=\frac{1}{2}\int_0^{\pi/6}\sin^2\theta\,\mathrm{d}\theta=\frac{1}{4}\int_0^{\pi/6}(1-\cos 2\theta)\,\mathrm{d}\theta$	M1	Formula for area applied to at least one curve ignoring limits	
				DM1	Deal with $\sin^2\theta$ correctly	
			$=\frac{1}{4}\left(\theta-\frac{1}{2}\sin 2\theta\right)$ $\left(=\frac{1}{4}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)\right)$	A1	answer before limits	
			For second: $A=\frac{1}{2}\int_{\pi/6}^{\pi/4} r^2\,\mathrm{d}\theta=\frac{1}{2}\int_{\pi/6}^{\pi/4}\cos^2 2\theta\,\mathrm{d}\theta=\frac{1}{4}\int_{\pi/6}^{\pi/4}(1+\cos 4\theta)\,\mathrm{d}\theta$	DM1	deal with $\cos^2 2\theta$ correctly	
			$=\frac{1}{4}\left(\theta+\frac{1}{4}\sin 4\theta\right)$ $\left(=\frac{\pi}{48}-\frac{\sqrt{3}}{32}\right)$	A1	answer before limits	
			Total area = $=\frac{\pi}{16}-\frac{3\sqrt{3}}{32}$	M1	Add *their* numerical areas from different limits (or subtract appropriately)	
				A1	cao	
				[8]		

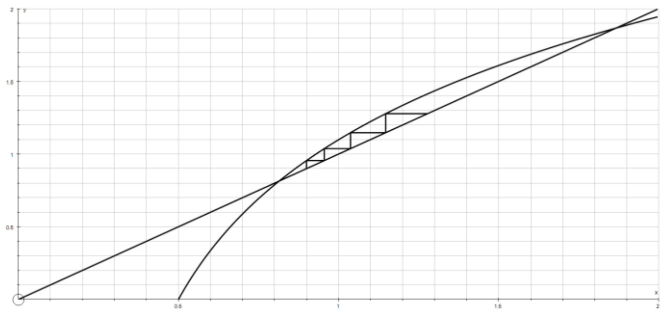
Question			Answer	Marks	Guidance	
6	(i)		$\sinh x = \frac{e^x - e^{-x}}{2}$, $\sinh 3x = \frac{e^{3x} - e^{-3x}}{2}$ $\sinh^3 x = \left(\frac{e^x - e^{-x}}{2}\right)^3 = \frac{1}{8}\left(e^{3x} - 3e^x + 3e^{-x} - e^{-3x}\right)$ $\Rightarrow 4\sinh^3 x = \left(\frac{e^{3x} - e^{-3x}}{2}\right) - 3\left(\frac{e^x - e^{-x}}{2}\right)$ $= \sinh 3x - 3\sinh x$ $\Rightarrow \sinh 3x = 4\sinh^3 x + 3\sinh x$	M1 A1 A1	Using exponentials for sinh$x$ and expanding cubic Correct cubic **oe** Must include exponential form of sinh3x	
				[3]		
	(ii)		$4w^3 + 3w - 3 = 0$ $\Rightarrow 4\sinh^3 x + 3\sinh x - 3 = 0$ $\Rightarrow \sinh 3x = 3$ $\Rightarrow 3x = \ln\left(3 \pm \sqrt{10}\right)$ $\Rightarrow x = \frac{1}{3}\ln\left(3 + \sqrt{10}\right)$ $\Rightarrow \left(w = \sinh\left(\frac{1}{3}\ln\left(3 + \sqrt{10}\right)\right)\right)$	M1 M1 A1 A1	Make substitution Use result of (i) Must have plus sign. Ignore subsequent working	
				[4]		

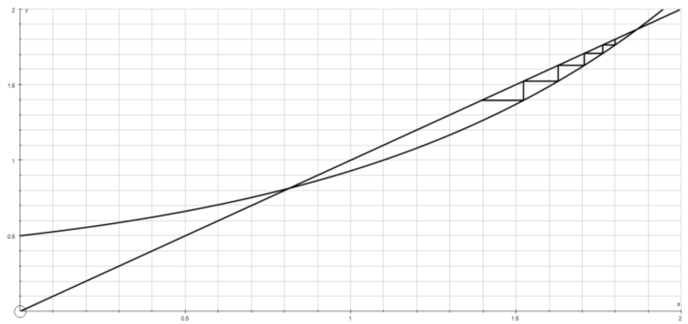
Question			Answer	Marks	Guidance	
7	(i)		$u = x^n \Rightarrow \mathrm{d}u = nx^{n-1}\mathrm{d}x$ $\mathrm{d}v = \sqrt{1-x}\ \mathrm{d}x \Rightarrow v = -\frac{2}{3}(1-x)^{3/2}$ $\Rightarrow I_n = \left[-x^n\frac{2}{3}(1-x)^{3/2}\right]_0^1 + \frac{2}{3}n\int_0^1 x^{n-1}(1-x)^{3/2}\,\mathrm{d}x$ $\Rightarrow I_n = 0 + \frac{2}{3}n\int_0^1 x^{n-1}(1-x)\sqrt{1-x}\,\mathrm{d}x$ $= \frac{2}{3}n(I_{n-1} - I_n)$ $\Rightarrow \left(1+\frac{2}{3}n\right)I_n = \frac{2n}{3}I_{n-1} \Rightarrow I_n = \frac{2n}{2n+3}I_{n-1}$	M1 A1 A1 M1 A1	Integrate by parts 1st term must be seen and later made = 0 Second term Dealing with $(1-x)^{3/2} = (1-x)\sqrt{1-x}$ And converting to I_n and I_{n-1} AG	
				[5]		
	(ii)		$\frac{2n}{2n+3} < 1 \Rightarrow I_n < I_{n-1}$	M1 A1		
				[2]		
	(iii)		$I_4 = \frac{8}{11}I_3 = \frac{8}{11}.\frac{6}{9}I_2 = \frac{8}{11}.\frac{6}{9}.\frac{4}{7}I_1 = \frac{8}{11}.\frac{6}{9}.\frac{4}{7}.\frac{2}{5}I_0$ $I_0 = \int_0^1 \sqrt{1-x}\,\mathrm{d}x = \left[-\frac{2}{3}(1-x)^{3/2}\right]_0^1 = \frac{2}{3}$ $\Rightarrow I_4 = \frac{8}{11}.\frac{6}{9}.\frac{4}{7}.\frac{2}{5}.\frac{2}{3} = \frac{256}{3465}$	M1 B1 A1	For using reduction formula For I_0 AG	$I_0 = \frac{2}{3},\ I_1 = \frac{4}{15},\ I_2 = \frac{16}{105},\ I_3 = \frac{32}{315}$ $\Rightarrow I_4 = \frac{256}{3465}$
				[3]		

Question			Answer	Marks	Guidance	
8	(i)		f(0.5) = –0.5 f(1) = 0.099 f(2) = –0.054 So sign changes gives roots in the required ranges. **oe**	**B1**	Must contain three decimal values and conclusion	e.g. "crosses axis"
				[1]		
	(ii)		$x_1 = 1.82455$ $x_2 = 1.84026$ $x_3 = 1.85019$ $\beta = 1.9$ is all that can be justified	**B1** **B1** **B1**	x_1 x_2 and x_3 Subtract 1 mark for not 5 dp β	
				[3]		
	(iii)		$x = \ln(4x-1) \Rightarrow 4x-1 = \mathrm{e}^x$ $\Rightarrow 4x = \mathrm{e}^x + 1 \Rightarrow x = \dfrac{\mathrm{e}^x+1}{4}$ giving $x_{r+1} = \dfrac{\mathrm{e}^{x_r}+1}{4}$ 0.81 0.811977 0.811977 0.813089 0.813089 0.813716 0.813716 0.814069 $\Rightarrow \alpha = 0.8145$	**M1** **A1** **M1** **A1**	Clear attempt to rearrange formula Must include the "*r*"s AG Starting with any value (which must be seen) in range [0.5,1] Mark final answer	
				[4]		

Question			Answer	Marks	Guidance	
	(iv)		For first iterative formula: $g(x)=\ln(4x-1)$ $\Rightarrow g'(x)=\dfrac{4}{4x-1}$ $\lvert g'(\alpha)\rvert>1$ (=1.77) so will not converge to α For 2nd iterative formula: $g(x)=\dfrac{e^x+1}{4}$ $\Rightarrow g'(x)=\dfrac{1}{4}e^x$ $\lvert g'(1.9)\rvert>1$ (=1.67) so it won't converge to β	**M1** **A1** **A1**	For finding g '(x) in either case. Or $\lvert g'(x)\rvert<1$ when $x>1.25$ so no Accept value in range [0.8,0.9] Or $\lvert g'(x)\rvert<1$ when $x<\ln 4\approx 1.4$ so no Accept value in range [1.8,2]	SC Demonstrating divergence numerically or graphically for either using an initial value near the root on either side. M1 1st demonstration plus conclusion (at least 3 numeric iterations or 2 steps)A1 2nd demonstration plus conclusion (at least 3 numeric iterations or 2 steps) A1
				[3]		

Graphs





Question			Answer	Marks	Guidance	
9	(i)		1st rectangle has area $\frac{1}{n}\mathrm{e}^{0}$ 2nd rectangle has area $\frac{1}{n}\mathrm{e}^{-\left(\frac{1}{n}\right)^2}$ Last rectangle has area $\frac{1}{n}\mathrm{e}^{-\left(\frac{n-1}{n}\right)^2}$ Giving $U=\frac{1}{n}\sum_{r=0}^{n-1}\mathrm{e}^{-\left(\frac{r}{n}\right)^2}$	M1 B1 A1	Evidence of using correct rectangles (may be by diagram) with heights the left hand side with a clear indication that they are all above the curve Correct width and height of at least one rectangle - do not accept $\frac{1}{n}\mathrm{e}^{-\left(\frac{0}{n}\right)^2}$ Considering last rectangle and sum giving complete solution www AG	SC: if n = 10 is used then M0 but B1 may be earned
				[3]		
	(ii)		$L=\frac{1}{n}\sum_{r=1}^{n}\mathrm{e}^{-\left(\frac{r}{n}\right)^2}$	B1		
				[1]		
	(iii)		$U-L=\frac{1}{n}\left(\sum_{r=0}^{n-1}\mathrm{e}^{-\left(\frac{r}{n}\right)^2}-\sum_{r=1}^{n}\mathrm{e}^{-\left(\frac{r}{n}\right)^2}\right)$ $=\frac{1}{n}\left(\mathrm{e}^{-\left(\frac{0}{n}\right)^2}-\mathrm{e}^{-\left(\frac{n}{n}\right)^2}\right)=\frac{1}{n}\left(1-\mathrm{e}^{-1}\right)$ $U-L<10^{-4}$ $\Rightarrow\frac{1}{n}\left(1-\mathrm{e}^{-1}\right)<10^{-4}$ $\Rightarrow n>\left(1-\mathrm{e}^{-1}\right)\times10^{4}\Rightarrow n=6322$	M1 A1 A1	Dealing with cancelling of terms Correct inequality soi	
				[3]		

OCR (Oxford Cambridge and RSA Examinations)
1 Hills Road
Cambridge
CB1 2EU

OCR Customer Contact Centre

Education and Learning
Telephone: 01223 553998
Facsimile: 01223 552627
Email: general.qualifications@ocr.org.uk

www.ocr.org.uk

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