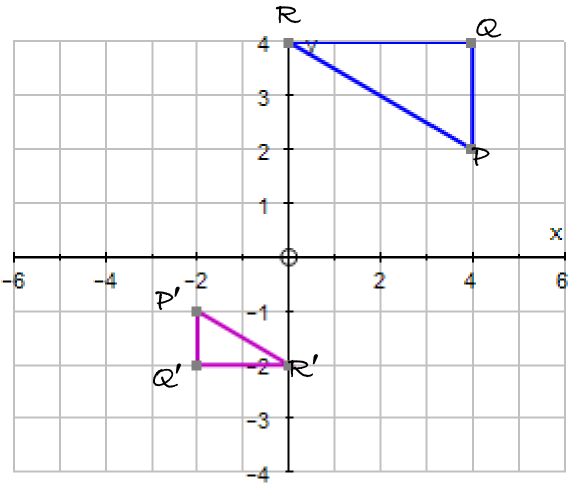
Section 2: Matrices and transformations

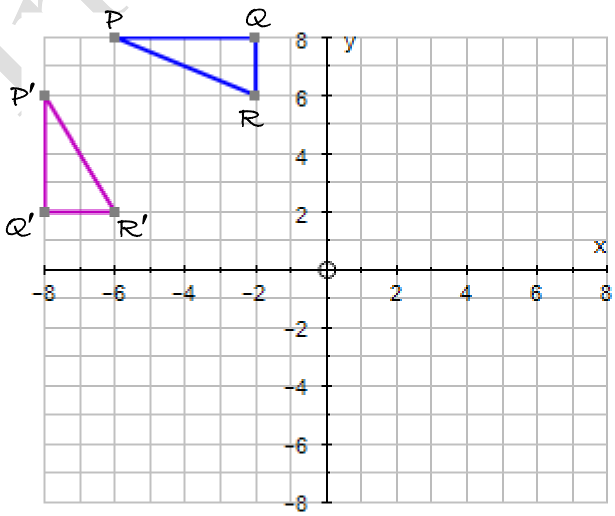
Solutions to Exercise level 2

1. (i) The image of P (4, 2) is P′ (-2, -1)
   The image of Q (4, 4) is Q′ (-2, -2)
   The image of R (0, 4) is R′ (0, -2)



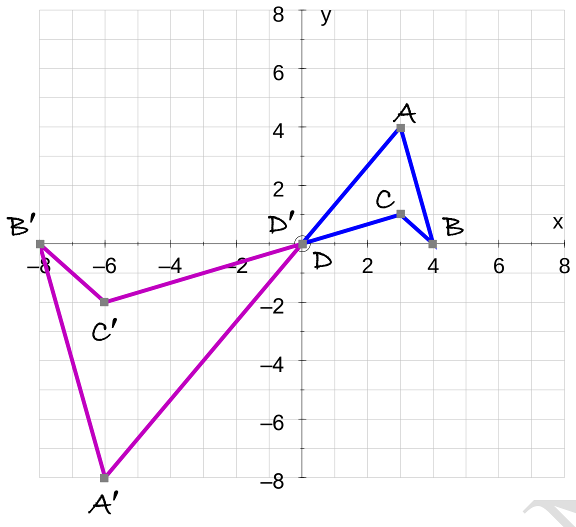
The transformation is an enlargement, scale factor -0.5, centre the origin.

(ii) The image of P (-6, 8) is P′ (-8, 6)
   The image of Q (-2, 8) is Q′ (-8, 2)
   The image of R (-2, 6) is R′ (-6, 2)



The transformation is a reflection in the line $y = -x$.

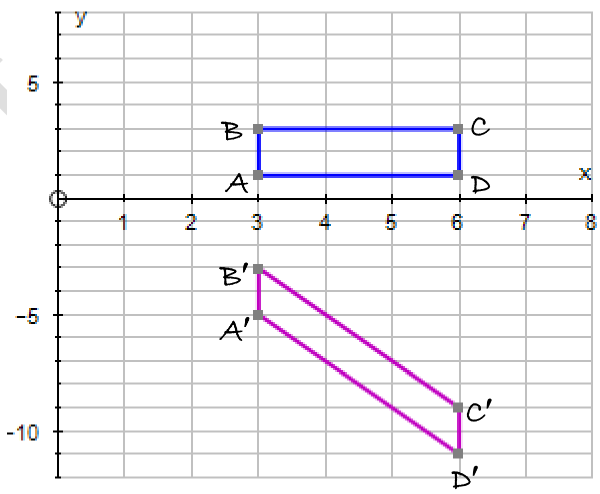
2. The image of the quadrilateral is A′ (-6, -8), B′ (-8, 0), C′(-6, -2), D′(0, 0).



The transformation is an enlargement, scale factor -2, centre the origin.

The area of the object $= 16 - 6 - 2 - 2 = 6$.
The area of the image $= 64 - 24 - 8 - 8 = 24$.
The ratio of the image area to the object area is 4 : 1.

3. The image of A (3, 1) is A′ (3, -5)
The image of B (3, 3) is B′ (3, -3)
The image of C (6, 3) is C′ (6, -9)
The image of D (6, 1) is D′ (6, -11).



The transformation is a shear with the y-axis fixed.

4. The matrix for an anticlockwise rotation through angle θ is
$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

so $R = \begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

5. (i) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6. (i) Comparing $\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$ with $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

gives $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = -\frac{1}{2}$

Since both are negative, θ must be in the 3rd quadrant

so $\theta = 180° + 30° = 210°$

Rotation through 210° anticlockwise (or 150° clockwise)

(ii) Comparing $\begin{pmatrix} -0.8 & -0.6 \\ 0.6 & -0.8 \end{pmatrix}$ with $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

gives $\cos\theta = -0.8$ and $\sin\theta = 0.6$

Since cos is negative and sin positive, θ must be in the 2nd quadrant

so $\theta = 180° - 36.9° = 143.1°$

Rotation through 143.1° anticlockwise.

7. Under P, the point (1, 0) is mapped to the point (-1, 0) and the point (0, 1) is unchanged.

So P is represented by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

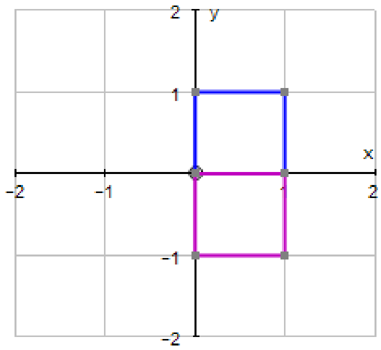
Under Q, the point (1, 0) is mapped to the point (0, -1) and the point (0, 1) is mapped to the point (1, 0).

So Q is represented by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The single matrix is $QP = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

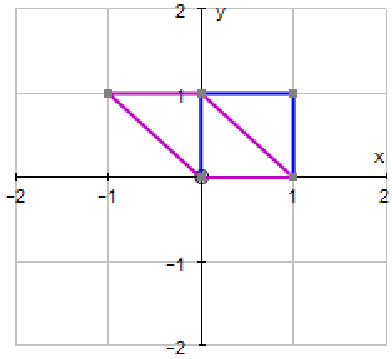
This transformation is a reflection in the line $y = x$.

8. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$



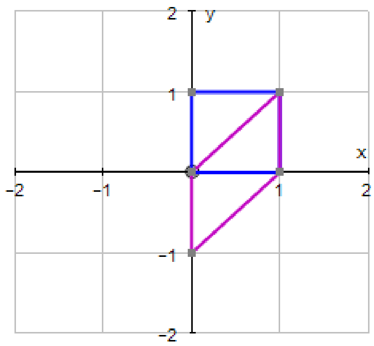
R is a rotation clockwise about the origin through 90°.

$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

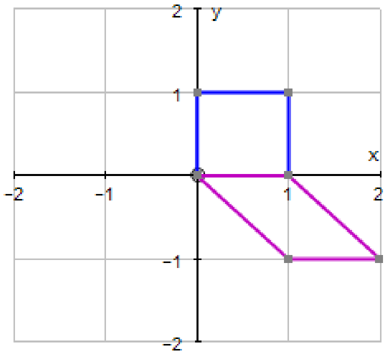


S is a shear parallel to the x-axis.

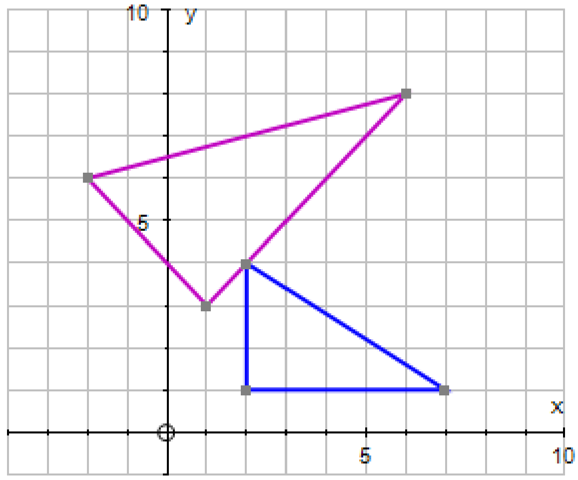
$RS = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$



$$SR = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$



9. (i), (ii) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 7 & 2 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 6 & -2 \\ 3 & 8 & 6 \end{pmatrix}$



(iii) Rotation anticlockwise through θ is represented by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Enlargement scale factor k is represented by $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Rotation followed by enlargement is represented by

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{pmatrix}$$

Comparing this with $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ gives:

$$\left.\begin{matrix} k\cos\theta = 1 \\ k\sin\theta = 1 \end{matrix}\right. \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45^\circ$$

$$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow k = \sqrt{2}$$

The angle of rotation is $45°$ anticlockwise and the scale factor of the enlargement is $\sqrt{2}$.

10. (i) $S = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) T is a shear with the x-axis fixed, with the point (0, 1) mapped to (2, 1).

(iii) $M = TS = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$