OCR RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4756

Further Methods for Advanced Mathematics (FP2)

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Friday 5 June 2009
Afternoon

Duration: 1 hour 30 minutes

4756

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Section A (54 marks)

Answer all the questions

1 **(a)** **(i)** Use the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$ to obtain the first three non-zero terms in the Maclaurin series for $\ln\left(\frac{1+x}{1-x}\right)$. State the range of validity of this series. **[4]**

(ii) Find the value of x for which $\frac{1+x}{1-x} = 3$. Hence find an approximation to $\ln 3$, giving your answer to three decimal places. **[4]**

(b) A curve has polar equation $r = \frac{a}{1+\sin\theta}$ for $0 \leqslant \theta \leqslant \pi$, where a is a positive constant. The points on the curve have cartesian coordinates x and y.

(i) By plotting suitable points, or otherwise, sketch the curve. **[3]**

(ii) Show that, for this curve, $r + y = a$ and hence find the cartesian equation of the curve. **[5]**

2 **(i)** Obtain the characteristic equation for the matrix $\mathbf{M}$ where

$$\mathbf{M} = \begin{pmatrix} 3 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Hence or otherwise obtain the value of $\det(\mathbf{M})$. **[3]**

(ii) Show that -1 is an eigenvalue of $\mathbf{M}$, and show that the other two eigenvalues are not real.

Find an eigenvector corresponding to the eigenvalue -1.

Hence or otherwise write down the solution to the following system of equations. **[9]**

$$\begin{aligned} 3x + y - 2z &= -0.1 \\ -y \quad &= 0.6 \\ 2x \quad + z &= 0.1 \end{aligned}$$

(iii) State the Cayley-Hamilton theorem and use it to show that

$$\mathbf{M}^3 = 3\mathbf{M}^2 - 3\mathbf{M} - 7\mathbf{I}.$$

Obtain an expression for $\mathbf{M}^{-1}$ in terms of $\mathbf{M}^2$, $\mathbf{M}$ and $\mathbf{I}$. **[4]**

(iv) Find the numerical values of the elements of $\mathbf{M}^{-1}$, showing your working. **[3]**

3 **(a)** **(i)** Sketch the graph of $y = \arcsin x$ for $-1 \leqslant x \leqslant 1$. **[1]**

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, justifying the sign of your answer by reference to your sketch. **[4]**

(ii) Find the exact value of the integral $\displaystyle\int_0^1 \frac{1}{\sqrt{2 - x^2}}\,\mathrm{d}x$. **[3]**

(b) The infinite series C and S are defined as follows.

$$C = \cos\theta + \tfrac{1}{3}\cos 3\theta + \tfrac{1}{9}\cos 5\theta + \ldots$$

$$S = \sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{9}\sin 5\theta + \ldots$$

By considering $C + \mathrm{j}S$, show that

$$C = \frac{3\cos\theta}{5 - 3\cos 2\theta},$$

and find a similar expression for S. **[11]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Prove, from definitions involving exponentials, that

$$\cosh 2u = 2\cosh^2 u - 1.$$ **[3]**

(ii) Prove that $\operatorname{arsinh} y = \ln\left(y + \sqrt{y^2 + 1}\right)$. **[4]**

(iii) Use the substitution $x = 2\sinh u$ to show that

$$\int \sqrt{x^2 + 4}\,\mathrm{d}x = 2\operatorname{arsinh}\tfrac{1}{2}x + \tfrac{1}{2}x\sqrt{x^2 + 4} + c,$$

where c is an arbitrary constant. **[6]**

(iv) By first expressing $t^2 + 2t + 5$ in completed square form, show that

$$\int_{-1}^{1} \sqrt{t^2 + 2t + 5}\,\mathrm{d}t = 2\left(\ln(1 + \sqrt{2}) + \sqrt{2}\right).$$ **[5]**

[Question 5 is printed overleaf.]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 Fig. 5 shows a circle with centre C $(a, 0)$ and radius a. B is the point $(0, 1)$. The line BC intersects the circle at P and Q; P is above the x-axis and Q is below.

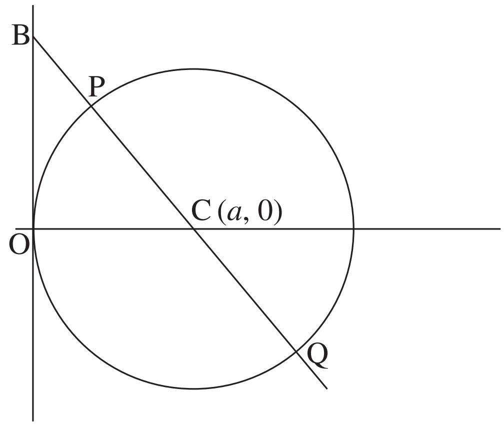

Fig. 5

(i) Show that, in the case $a = 1$, P has coordinates $\left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Write down the coordinates of Q. **[3]**

(ii) Show that, for all positive values of a, the coordinates of P are

$$x = a\left(1 - \frac{a}{\sqrt{a^2+1}}\right), \qquad y = \frac{a}{\sqrt{a^2+1}}. \qquad (*)$$

Write down the coordinates of Q in a similar form. **[4]**

Now let the variable point P be defined by the parametric equations $(*)$ for all values of the parameter a, positive, zero and negative. Let Q be defined for all a by your answer in part **(ii)**.

(iii) Using your calculator, sketch the locus of P as a varies. State what happens to P as $a \to \infty$ and as $a \to -\infty$.

Show algebraically that this locus has an asymptote at $y = -1$.

On the same axes, sketch, as a dotted line, the locus of Q as a varies. **[8]**

(The single curve made up of these two loci and including the point B is called a *right strophoid*.)

(iv) State, with a reason, the size of the angle POQ in Fig. 5. What does this indicate about the angle at which a right strophoid crosses itself? **[3]**