GCE

Mathematics

Advanced GCE

Unit **4727:** Further Pure Mathematics 3

Mark Scheme for June 2011

Oxford Cambridge and RSA Examinations

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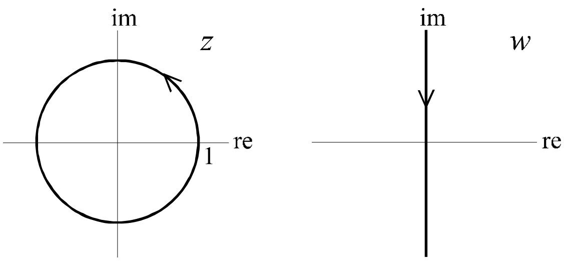
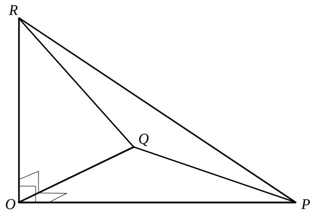
Question	Answer	Marks	Guidance
1 (i)	$\theta = \sin^{-1}\dfrac{\lvert[5, 6, -7]\cdot[1, 2, -1]\rvert}{\sqrt{5^2+6^2+(-7)^2}\sqrt{1^2+2^2+(-1)^2}}$	M1*	For using scalar product of line and plane vectors
		M1 (*dep)	For both moduli seen
	$\theta = \sin^{-1}\dfrac{24}{\sqrt{110}\sqrt{6}} = 69.1^\circ$ $(69.099...^\circ, 1.206)$	A1	For correct scalar product
		A1 **4**	For correct angle
	$\phi = \sin^{-1}\dfrac{\lvert[5, 6, -7]\times[1, 2, -1]\rvert}{\sqrt{5^2+6^2+(-7)^2}\sqrt{1^2+2^2+(-1)^2}}$	**SR** M1*	For vector product of line and plane vectors AND finding modulus of result
		M1 (*dep)	For moduli of line and plane vectors seen
	$\phi = \sin^{-1}\dfrac{\sqrt{84}}{\sqrt{110}\sqrt{6}} = 20.9^\circ \Rightarrow \theta = 69.1^\circ$	A1	For correct modulus $\sqrt{84}$
		A1	For correct angle
(ii)	METHOD 1		
	$d = \dfrac{\lvert 1+12+3-40\rvert}{\sqrt{1^2+2^2+(-1)^2}} = \dfrac{24}{\sqrt{6}} = 4\sqrt{6} \approx 9.80$	M1	For use of correct formula
		A1 **2**	For correct distance
	METHOD 2		
	$(1+\lambda)+2(6+2\lambda)-(-3-\lambda)=40$	M1	For substituting parametric form into plane
	$\Rightarrow \lambda = 4 \Rightarrow d = 4\sqrt{6}$	A1	For correct distance
	OR distance from $(1, 6, -3)$ to $(5, 14, -7)$ $= \sqrt{4^2+8^2+(-4)^2} = \sqrt{96}$		
	METHOD 3		
	Plane through $(1, 6, -3)$ parallel to p is	M1	For finding parallel plane through $(1, 6, -3)$
	$x+2y-z=16 \Rightarrow d = \dfrac{40-16}{\sqrt{6}} = \dfrac{24}{\sqrt{6}}$	A1	For correct distance
	METHOD 4		
	e.g. $(0, 0, -40)$ on p $\Rightarrow$ vector to $(1, 6, -3) = \pm(1, 6, 37)$	M1	For using any point on p to find vector and scalar product seen e.g. $[1, 6, 37]\cdot[1, 2, -1]$
	$d = \dfrac{\lvert[1, 6, 37]\cdot[1, 2, -1]\rvert}{\sqrt{6}} = \dfrac{24}{\sqrt{6}}$	A1	For correct distance
	METHOD 5		
	l meets p where $(1+5t)+2(6+6t)-(-3-7t)=40$ $\Rightarrow t=1 \Rightarrow d = \lvert[5, 6, -7]\rvert\sin\theta$	M1	For finding t where l meets p and linking d with triangle
	$\Rightarrow d = \sqrt{110}\dfrac{24}{\sqrt{110}\sqrt{6}} = \dfrac{24}{\sqrt{6}}$	A1	For correct distance
		6	
2 (i)	METHOD 1		
	EITHER $\dfrac{1+e^{i\theta}}{1-e^{i\theta}} = \dfrac{e^{-\frac{1}{2}i\theta}+e^{\frac{1}{2}i\theta}}{e^{-\frac{1}{2}i\theta}-e^{\frac{1}{2}i\theta}}$	M1	*EITHER* For changing LHS terms to $e^{\pm\frac{1}{2}i\theta}$ *OR in reverse* For using $\cot\frac{1}{2}\theta = \dfrac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$
	$= \dfrac{2\cos\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta} = i\cot\frac{1}{2}\theta$	M1	For either of $\frac{\cos}{\sin}\frac{1}{2}\theta = \dfrac{e^{\frac{1}{2}i\theta} \pm e^{-\frac{1}{2}i\theta}}{(2)(i)}$ soi
	OR in reverse with similar working	A1 **3**	For fully correct proof to **AG** **SR** If factors of 2 or i are not clearly seen, award M1 M1 A0

2 (i)	METHOD 2			
	EITHER $\dfrac{1+\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}} \times \dfrac{1-\mathrm{e}^{-\mathrm{i}\theta}}{1-\mathrm{e}^{-\mathrm{i}\theta}} = \dfrac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}}{2-\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)}$	M1		For multiplying top and bottom by complex conjugate in exp or trig form
	OR $\dfrac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta} \times \dfrac{1-\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta+\mathrm{i}\sin\theta}$			
	$= \dfrac{2\mathrm{i}\sin\theta}{2-2\cos\theta} = \dfrac{2\mathrm{i}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = \mathrm{i}\cot\frac{1}{2}\theta$	M1 A1		For using both double angle formulae correctly For fully correct proof to **AG**
	METHOD 3			
	$\dfrac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta} = \dfrac{2\cos^2\frac{1}{2}\theta+2\mathrm{i}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta-2\mathrm{i}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$	M1		For using both double angle formulae correctly
	$= \dfrac{2\cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta+\mathrm{i}\sin\frac{1}{2}\theta\right)}{2\sin\frac{1}{2}\theta\left(\sin\frac{1}{2}\theta-\mathrm{i}\cos\frac{1}{2}\theta\right)}$	M1		For appropriate factorisation
	$= \mathrm{i}\cot\frac{1}{2}\theta\,\dfrac{\left(\sin\frac{1}{2}\theta-\mathrm{i}\cos\frac{1}{2}\theta\right)}{\left(\sin\frac{1}{2}\theta-\mathrm{i}\cos\frac{1}{2}\theta\right)} = \mathrm{i}\cot\frac{1}{2}\theta$	A1		For fully correct proof to **AG**
	METHOD 4			
	$\dfrac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta} = \dfrac{1+\dfrac{1-t^2}{1+t^2}+\mathrm{i}\dfrac{2t}{1+t^2}}{1-\dfrac{1-t^2}{1+t^2}-\mathrm{i}\dfrac{2t}{1+t^2}}$	M1		For substituting both t formulae correctly
	$= \dfrac{2+2\mathrm{i}t}{2t^2-2\mathrm{i}t} = \dfrac{1}{t}\dfrac{1+\mathrm{i}t}{t-\mathrm{i}} = \dfrac{\mathrm{i}}{t}\dfrac{t-\mathrm{i}}{t-\mathrm{i}} = \mathrm{i}\cot\frac{1}{2}\theta$	M1 A1		For appropriate factorisation For fully correct proof to **AG**
	METHOD 5			
	$\dfrac{1+\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}} \times \dfrac{1+\mathrm{e}^{\mathrm{i}\theta}}{1+\mathrm{e}^{\mathrm{i}\theta}} = \dfrac{1+2\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{2\mathrm{i}\theta}}{1-\mathrm{e}^{2\mathrm{i}\theta}}$ $= \dfrac{2+\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}}{\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}}$	M1		For multiplying top and bottom by $1+\mathrm{e}^{\mathrm{i}\theta}$ and attempting to divide by $\mathrm{e}^{\mathrm{i}\theta}$ *OR* multiplying top and bottom by $1+\mathrm{e}^{-\mathrm{i}\theta}$
	$= \dfrac{2(1+\cos\theta)}{-2\mathrm{i}\sin\theta} = \dfrac{2\cos^2\frac{1}{2}\theta}{-2\mathrm{i}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \dfrac{\cos\frac{1}{2}\theta}{-\mathrm{i}\sin\frac{1}{2}\theta}$	M1		For using both double angle formulae correctly
	$= \mathrm{i}\cot\frac{1}{2}\theta$	A1	3	For fully correct proof to **AG**
(ii)		M1 A1 B1	 3	For a circle centre O For indication of radius = 1 and anticlockwise arrow shown For locus of w shown as imaginary axis described downwards
			6	

3 (i)	METHOD 1 $m+4\ (=0) \Rightarrow$ CF $(y=)Ae^{-4x}$	M1 A1	**2**	For correct auxiliary equation (soi) For correct CF
	METHOD 2 Separating variables on $\frac{dy}{dx}+4y=0$ $\Rightarrow \ln y = -4x$ $\Rightarrow$ CF $(y=)Ae^{-4x}$	M1 A1		For integration to this stage For correct CF
(ii)	PI $(y=)\ p\cos 3x + q\sin 3x$ $y' = -3p\sin 3x + 3q\cos 3x$ $\Rightarrow (-3p+4q)\sin 3x + (4p+3q)\cos 3x = 5\cos 3x$ $\Rightarrow \left.\begin{matrix}-3p+4q=0\\ 4p+3q=5\end{matrix}\right\} \Rightarrow p=\frac{4}{5},\ q=\frac{3}{5}$ GS $(y=)\ Ae^{-4x}+\frac{4}{5}\cos 3x+\frac{3}{5}\sin 3x$	B1 M1 A1 M1 A1 A1 B1√	**7**	For stating PI of correct form For substituting y and y' into DE For correct equation For equating coeffs and solving For correct value of p, and of q For GS f.t. from their CF+PI with 1 arbitrary constant in CF and none in PI
	SR Integrating factor method may be used, followed by 2-stage integration by parts or C+iS method Marks for **(i)** are awarded only if CF is clearly identified			
(iii)	$e^{-4x} \to 0$, $\frac{4}{5}\cos 3x+\frac{3}{5}\sin 3x = \frac{\sin}{\cos}(3x+\alpha)$ $\Rightarrow\ -1 \leqslant y \leqslant 1$ *OR* $-1 \lessapprox y \lessapprox 1$	M1 A1√	**2**	For considering either term For correct range (allow <) **CWO** f.t. as $-\sqrt{p^2+q^2} \leqslant y \leqslant \sqrt{p^2+q^2}$ from **(ii)**
			11	
4 (i)	$abc=(ab)c=(ba)c=b(ac)=$ $b(ca)=(bc)a=(cb)a=cba$ Minimum working: $abc=bac=bca=cba$ *OR* $abc=acb=cab=cba$ *OR* $abc=bac=bca=cba$	M1 A1	**2**	For using commutativity correctly For correct proof (use of associativity may be implied)
(ii)	$\{e, a\}, \{e, b\}, \{e, c\}, \{e, bc\}, \{e, ca\}, \{e, ab\}, \{e, abc\}$	B1 B1	**2**	For any 5 subgroups For the other 2 subgroups and none incorrect
(iii)	$\{e, a, b, ab\}, \{e, a, c, ca\}, \{e, b, c, bc\}$ $\{e, a, bc, abc\}, \{e, b, ca, abc\}, \{e, c, ab, abc\}$ $\{e, bc, ca, ab\}$	B1 B1 B1	**3**	For any 3 subgroups For 1 more subgroup For 1 more subgroup (5 in total) and none incorrect
(iv)	All elements $(\neq e)$ have order 2 *OR* all are self-inverse *OR* no element of G has order 4 *OR* no order 4 subgroup has a generator *or* is cyclic *OR* subgroups are of the form $\{e, a, b, ab\}$ (the Klein group) $\Rightarrow$ all order 4 subgroups are isomorphic	B1* B1 (*dep)	**2**	For appropriate reference to order of elements in G For correct conclusion
			9	

5 (i)	$\frac{dy}{dx} = k u^{k-1} \frac{du}{dx}$	M1 A1		For using chain rule For correct $\frac{dy}{dx}$
	$\Rightarrow x k u^{k-1} \frac{du}{dx} + 3u^k = x^2 u^{2k}$	M1		For substituting for y and $\frac{dy}{dx}$
	$\Rightarrow \frac{du}{dx} + \frac{3}{kx} u = \frac{1}{k} x u^{k+1}$	A1	4	For correct equation **AG**
(ii)	$k = -1$	B1	1	For correct k
(iii)	$\frac{du}{dx} - \frac{3}{x} u = -x \Rightarrow$ IF $e^{-\int \frac{3}{x} dx} = e^{-3\ln x} = \frac{1}{x^3}$	B1√		For correct IF f.t. for IF = $x^{\frac{3}{k}}$ using k or their numerical value for k
	$\Rightarrow \frac{d}{dx}\left(u . \frac{1}{x^3}\right) = -\frac{1}{x^2}$	M1		For $\frac{d}{dx}(u.\text{their IF}) = -x.\text{their IF}$
	$\Rightarrow u . \frac{1}{x^3} = \frac{1}{x} (+c) \Rightarrow y = \frac{1}{cx^3 + x^2}$	A1 A1	4	For correct integration both sides For correct solution for y
			9	
6 (a)	Closure $(ax+b)+(cx+d)=(a+c)x+(b+d)$	B1		For obtaining correct sum from 2 distinct elements
	$\in P$	B1		For stating result is in P *OR* is of the correct form **SR** award this mark if any of the closure result, the identity or the inverse element is stated to be in P *OR* of the correct form
	Identity $0x+0$	B1		For stating identity (allow 0)
	Inverse $-ax-b$	B1	4	For stating inverse
(b) (i)	Order 9	B1*	1	For correct order
(ii)	$x+2$	B1	1	For correct inverse element
(iii)	$(ax+b)+(ax+b)+(ax+b)=3ax+3b$	M1		For considering sums of $ax+b$ and obtaining $3ax+3b$
	$= 0x+0$ $\Rightarrow ax+b$ has order 3 $\forall\ a, b$ (except $a=b=0$)	A1		For equating to $0x+0$ *OR* 0 and obtaining order 3 **SR** For order 3 stated only *OR* found from incomplete consideration of numerical cases award B1
	Cyclic group of order 9 has element(s) of order 9	M1 (*dep)		For reference to element(s) of order 9
	$\Rightarrow (Q, +(\text{mod } 3))$ is not cyclic	A1	4	For correct conclusion
			10	

7 (i)		B1		For sketch of tetrahedron labelled in some way At least one right angle at *O* must be indicated or clearly implied
		M1		For using $\Delta = \frac{1}{2}$ base × height
	$\Delta\ OPQ = \frac{1}{2}pq$, $\Delta\ OQR = \frac{1}{2}qr$, $\Delta\ ORP = \frac{1}{2}rp$	A1	3	For all areas correct **CAO**
(ii)	$\frac{1}{2}\left\|\overrightarrow{RP} \times \overrightarrow{RQ}\right\| = \frac{1}{2}\|\overrightarrow{RP}\|\|\overrightarrow{RQ}\|\sin R = \Delta\ PQR$	B1	1	For correct justification
(iii)	LHS $= \left(\frac{1}{2}pq\right)^2 + \left(\frac{1}{2}qr\right)^2 + \left(\frac{1}{2}rp\right)^2$	B1		For correct expression
	$\Delta\ PQR = \frac{1}{2}\left\|(p\mathbf{i} - q\mathbf{j}) \times (p\mathbf{i} - r\mathbf{k})\right\|$ *OR* $\frac{1}{2}\left\|(p\mathbf{i} - r\mathbf{k}) \times (q\mathbf{j} - r\mathbf{k})\right\|$ *OR* $\frac{1}{2}\left\|(p\mathbf{i} - q\mathbf{j}) \times (q\mathbf{j} - r\mathbf{k})\right\|$	B1		For ΔPQR in vector form
	$\Delta\ PQR = \frac{1}{2}\left\|qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k}\right\|$	M1		For finding vector product of their attempt at ΔPQR
		A1		For correct expression
	RHS $= \frac{1}{4}\left((pq)^2 + (qr)^2 + (rp)^2\right)$	M1		For using $\left\|a\mathbf{i} + b\mathbf{j} + c\mathbf{k}\right\| = \sqrt{a^2 + b^2 + c^2}$
		A1	6	For completing proof of **AG WWW**
			10	

Question	Answer	Marks	Guidance
8 (i)	$\text{Re}(c+\text{i}\,s)^4 = \cos 4\theta = c^4 - 6c^2s^2 + s^4$	M1*	For expanding $(c+\text{i}\,s)^4$: at least 2 terms and 1 binomial coefficient needed
		A1	For 3 correct terms
	$\cos 4\theta = c^4 - 6c^2(1-c^2) + (1-c^2)^2$	M1 (*dep)	For using $s^2 = 1 - c^2$
	$\Rightarrow \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$	A1 **4**	For correct expression for $\cos 4\theta$ **CAO**
(ii)	$\cos 4\theta \cos 2\theta = \left(8c^4 - 8c^2 + 1\right)\left(2c^2 - 1\right)$		For multiplying by $\left(2c^2 - 1\right)$
	$= 16\cos^6\theta - 24\cos^4\theta + 10\cos^2\theta - 1$	B1 **1**	to obtain **AG WWW**
(iii)	$16c^6 - 24c^4 + 10c^2 - 2 = 0$	M1	For factorising sextic
	$\Rightarrow \left(c^2 - 1\right)\left(8c^4 - 4c^2 + 1\right) = 0$		with $(c-1), (c+1)$ *or* $(c^2 - 1)$
	For quartic, $b^2 - 4ac = 16 - 32 < 0$	A1	For justifying no other roots **CWO**
	$\Rightarrow c = \pm 1$ only $\Rightarrow \theta = n\pi$	A1 **3**	For obtaining $\theta = n\pi$ **AG**
			Note that M1 A0 A1 is possible
		SR	For verifying $\theta = n\pi$ by substituting $c = \pm 1$ into $16c^6 - 24c^4 + 10c^2 - 2 = 0$ B1
(iv)	$16c^6 - 24c^4 + 10c^2 = 0$		
	$\Rightarrow c^2\left(8c^4 - 12c^2 + 5\right) = 0$	M1	For factorising sextic with c^2
	For quartic, $b^2 - 4ac = 144 - 160 < 0$	A1	For justifying no other roots **CWO**
	$\Rightarrow \cos\theta = 0$ only	A1 **3**	For correct condition obtained **AG**
			Note that M1 A0 A1 is possible
		SR	For verifying $\cos\theta = 0$ by substituting $c = 0$ into $16c^6 - 24c^4 + 10c^2 = 0$ B1
		SR	For verifying $\theta = \frac{1}{2}\pi$ and $\theta = -\frac{1}{2}\pi$ satisfy $\cos 4\theta \cos 2\theta = -1$ B1
		11	

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