Qualification Accredited

Oxford Cambridge and RSA

A LEVEL

Examiners' report

FURTHER MATHEMATICS B (MEI)

H645
For first teaching in 2017

Y420/01 Summer 2019 series

Version 1

Contents

Introduction

Our examiners' reports are produced to offer constructive feedback on candidates' performance in the examinations. They provide useful guidance for future candidates. The reports will include a general commentary on candidates' performance, identify technical aspects examined in the questions and highlight good performance and where performance could be improved. The reports will also explain aspects that caused difficulty and why the difficulties arose, whether through a lack of knowledge, poor examination technique, or any other identifiable and explainable reason.

Where overall performance on a question/part was considered good, with no particular areas to highlight, these questions have not been included in the report. A full copy of the question paper can be downloaded from OCR.

Paper Y420 series overview

This year was the first setting of this paper. The paper is relatively long and presents a challenging test to candidates, however it was encouraging to see how many responded with sustained, excellent work. Approximately half of the candidates scored over 100 marks, however there was also a significant 'tail' of scores of 60 marks or less.

The standard of presentation varied enormously from extremely well-presented scripts to ones that were extremely difficult to decipher or follow, especially in the high tariff questions.

	AfL	It is recommended that candidates are given practice at tackling unstructured problems (such as this paper's question 12) that require careful structuring of solutions. Often in responses the correct answer would appear without sufficient detail of working to justify the award of the marks.

	AfL	Section A is designed to provide some relatively straightforward assessment and all candidates were able to demonstrate their knowledge and skills with some or all of these short questions. Section B tested candidates with longer questions involving problem solving (some unstructured) and performance on these questions clearly differentiated candidates, with high scores and low scores and fewer in the middle. The majority of candidates managed to respond to the later questions, although there was evidence of some candidates running out of time.

	AfL	There were occurrences of candidates not showing sufficient working in 'Direct Reasoning' questions. This command is included principally in questions where a modern calculator can supply answers automatically without the need for 'traditional' techniques such as solving equations by formula or factorising, rationalising or realising denominators, integration, etc. While it is quite possible for candidates to check their answer(s) to these questions using their calculator (and indeed it is sensible to encourage this) it is important that they provide working, demonstrating their knowledge of mathematical techniques in their solutions.

Questions 2, 3, 4, 5, 7, 8, 10(a), 11(a), 14(a), 16(a) and 17(c) were generally well answered.

Questions 13, 14(b), 15, 16(b) and 17(f) proved to be the most demanding.

Section A overview

This section proved to be accessible to most candidates and many scored well on these relatively straightforward tests.

Question 1

1 Find $\sum_{r=1}^{n}(2r^2-1)$, expressing your answer in fully factorised form. **[4]**

Most candidates substituted the standard sum of the series Σr^2 and used $\sum_{r=1}^{n}1=n$. Quite a few however then went on to make errors with the subsequent algebra. Just over 60% of candidates gained full marks.

Question 2

2 The plane $x+2y+cz=4$ is perpendicular to the plane $2x-cy+6z=9$, where c is a constant. Find the value of c. **[3]**

The scalar product was well known and approximately 80% of candidates scored full marks.

Question 3 (a)

3 Matrices **A** and **B** are defined by $\mathbf{A}=\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}k & 1\\ 2 & 0\end{pmatrix}$, where k is a constant.

(a) Verify the result $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$ in this case. **[5]**

This was usually well done. A few candidates however omitted the reciprocal of the determinant when finding inverses.

Question 3 (b)

(b) Investigate whether **A** and **B** are commutative under matrix multiplication. **[2]**

Most candidates calculated BA correctly and then just stated that A and B were not commutative, without finding the value of *k* for which they were commutative, i.e. *k* = 2 (as shown in Exemplar 1).

Exemplar 1

3(b) If the multiplication was commutative $AB = BA$

$$AB = \begin{pmatrix} 3k+2 & 3 \\ 2k+2 & 2 \end{pmatrix} \qquad BA = \begin{pmatrix} k & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3k+2 & k+1 \\ 6+0 & 2+0 \end{pmatrix} \neq AB$$

Therefore the multiplication is not commutative.

?	**Misconception**	Many candidates appear to have been taught that 'matrix multiplication is not commutative'. While this is true in general, it is worth emphasising that this is not necessarily true for specific matrices.

Question 4

4 In this question you must show detailed reasoning.

Fig. 4 shows the region bounded by the curve $y = \sec\frac{1}{2}x$, the x-axis, the y-axis and the line $x = \frac{1}{2}\pi$.

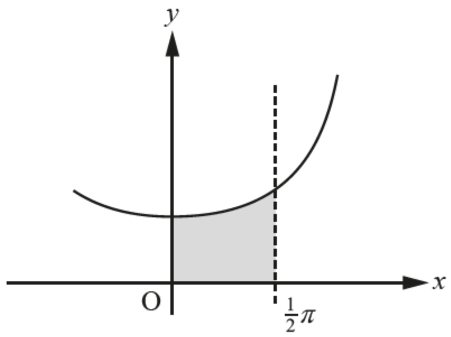


Fig. 4

This region is rotated through 2π radians about the x-axis.
Find, in exact form, the volume of the solid of revolution generated. **[3]**

Three quarters of the candidates scored full marks here, recognising the integral of $\sec^2 \frac{1}{2} x$ as $2\tan \frac{1}{2} x$.

Question 5

5 Using the Maclaurin series for $\cos 2x$, show that, for small values of x,

$$\sin^2 x \approx ax^2 + bx^4 + cx^6,$$

where the values of a, b and c are to be given in exact form. **[5]**

Some candidates derived the Maclaurin series for cos 2*x* from first principles, which cost them valuable time. The use of $\cos 2x = 1 - 2\sin^2 x$ was usually employed successfully. However, some candidates started with $\sin^2 x = \cos^2 x - \cos 2x$ and tried to square the Maclaurin series for cos *x*, sometimes without using enough terms.

Question 6

6 **In this question you must show detailed reasoning.**

Find $\displaystyle\int_2^{\infty} \frac{1}{4+x^2}\,dx$. **[4]**

The standard integral result was well known. Thereafter a good proportion of solutions employed the limit concept accurately to evaluate the improper integral (see Exemplar 2).

Exemplar 2

let ∞ be a

$$\int_2^a \frac{1}{4+x^2}\,dx = \left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_2^a$$

$$= \frac{1}{2}\arctan\left(\frac{a}{2}\right) - \frac{1}{2}\arctan 1$$

$$= \frac{1}{2}\arctan\left(\frac{a}{2}\right) - \frac{1}{8}\pi$$

when $a \to \infty$ $\arctan\left(\frac{a}{2}\right) \to \frac{1}{2}\pi$

$$\therefore \int_2^{\infty} \frac{1}{4+x^2}\,dx = \frac{1}{4}\pi - \frac{1}{8}\pi = \frac{1}{8}\pi$$

Simply stating the result without showing this limit lost a mark.

Question 7 (a)

7 A curve has cartesian equation $(x^2 + y^2)^2 = 2c^2xy$, where c is a positive constant.

(a) Show that the polar equation of the curve is $r^2 = c^2 \sin 2\theta$. **[2]**

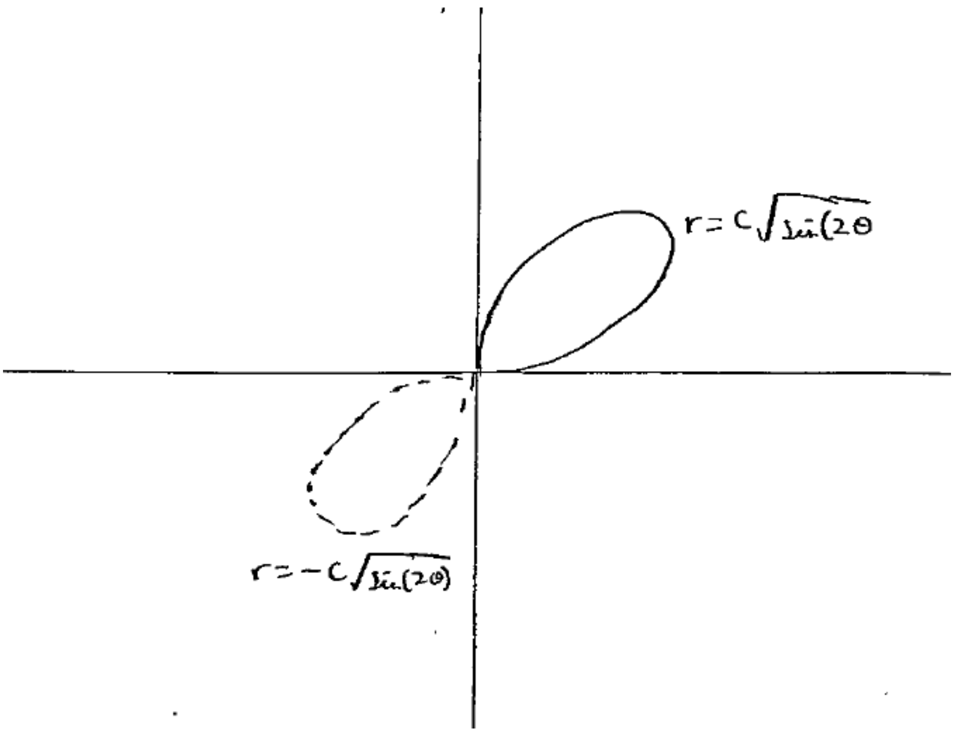
Approximately 80% of candidates gained full marks in this question, with most applying the conversion formulae correctly.

Question 7 (b)

(b) Sketch the curves $r = c\sqrt{\sin 2\theta}$ and $r = -c\sqrt{\sin 2\theta}$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. **[3]**

A mark here was awarded for using a dashed line for values of θ that yield a negative value of *r*, as in Exemplar 3.

Exemplar 3



This was lost in most responses, which scored 2 marks out of 3.

Question 7 (c)

(c) Find the area of the region enclosed by one of the loops in part **(b)**. **[3]**

This was straightforward and usually well done.

Section B overview

Candidate performance in this section proved to be much more variable. Some of these questions required a level of accuracy and resolve that was a challenge for candidates.

Question 8 (a)

8 In this question you must show detailed reasoning.

The roots of the equation $x^3 - x^2 + kx - 2 = 0$ are α, $\frac{1}{\alpha}$ and β.

(a) Evaluate, in exact form, the roots of the equation. **[6]**

The symmetric property of the roots was well understood and there were many successful attempts at this question. The final mark could not be awarded often however, since candidates had not clearly identified the roots. Some candidates verified that if $\alpha = (-1+\sqrt{3})/2$ then $1/\alpha = (-1-\sqrt{3})/2$, although in Standardisation it was decided to condone the omission of this step.

Question 8 (b)

(b) Find k. **[2]**

This was well done, either using the values of the roots or the factor theorem with $x = 2$.

Question 9

9 Prove by induction that $5^n + 2 \times 11^n$ is divisible by 3 for all positive integers n. **[7]**

The inductive step for divisibility proofs can be quite tricky and we found some interesting variations for negotiating this.

The structure of an inductive proof was generally well understood, and most candidates who overcame the inductive step gained full marks. We expect a careful summary statement of the inductive principle at the end to achieve the final 'A' mark, along the lines of 'as it is true for n = 1, and if true for n = k then true for n = k+1, it is therefore true for all n'.

	AfL	It is worth emphasising the technique of substituting for or extracting multiples of $3m - 5\times11^k$ for 5^k (or its equivalent) in induction proofs of divisibility like this.

Question 10 (a)

10 In this question you must show detailed reasoning.

(a) You are given that $-1+\mathrm{i}$ is a root of the equation $z^3 = a + b\mathrm{i}$, where a and b are real numbers. Find a and b. **[3]**

This was often well tackled, either using the binomial expansion or simply multiplying out. Some candidates however found –1 + i in modulus argument form and then used De Moivre's theorem to find its cube. Others stated that –1 –i was another root, notwithstanding part (c).

Question 10 (b)

(b) Find all the roots of the equation in part **(a)**, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r and θ are exact. **[4]**

Candidates in most cases scored 0 or 4. We accepted arguments within the range 0 to 2π or $-\pi$ to π. Errors in the modulus and argument of –1 + i proved rather costly.

Question 10 (c)

(c) Chris says "the complex roots of a polynomial equation come in complex conjugate pairs". Explain why this does **not** apply to the polynomial equation in part **(a)**. **[1]**

Quite a few candidates thought that this was to do with having an odd number of roots, rather than a complex coefficient.

Question 11 (a)

11 (a) Specify fully the transformations represented by the following matrices.

- $\mathbf{M}_1 = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

- $\mathbf{M}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **[4]**

This question was well answered. However, a number of candidates omitted to specify 'about the origin' for the rotation.

Question 11 (b)

(b) Find the equation of the mirror line of the reflection R represented by the matrix $\mathbf{M}_3 = \mathbf{M}_1\mathbf{M}_2$. **[5]**

Finding the line of invariant points was the common and most successful approach to this, even though occasionally we saw solutions that found the perpendicular bisector of a point of its image. Candidates attempting this often did not do this successfully or omitted the rejection of $y = -2x$.

?	**Misconception**	Teachers should emphasise the distinction between a line of invariant point and an invariant line.

Question 11 (c)

(c) It is claimed that the reflection represented by the matrix $\mathbf{M}_4 = \mathbf{M}_2\mathbf{M}_1$ has the same mirror line as R. Explain whether or not this claim is correct. **[3]**

If candidates rejected the claim because 'matrix multiplication is not commutative' then they scored no marks, on the grounds some matrices do commute. Some seemed know that reflection and rotation are not commutative, but they were required them to give some justification for this. The best responses were those who evaluated M_4 and found the mirror line for its transformation, however showing that M_1M_2 was different to M_2M_1, and the mirror lines must therefore differ was also awarded 3 marks.

Question 12

12 Three intersecting lines L_1, L_2 and L_3 have equations

$$L_1\colon \frac{x}{2} = \frac{y}{3} = \frac{z}{1}, \quad L_2\colon \frac{x}{1} = \frac{y}{2} = \frac{z}{-4} \quad \text{and} \quad L_3\colon \frac{x-1}{1} = \frac{y-2}{1} = \frac{z+4}{5}.$$

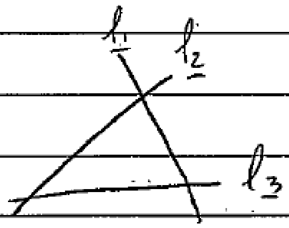
Find the area of the triangle enclosed by these lines. **[9]**

The most efficient solutions found the points of intersection of the lines, then either used the cross product to evaluate the area, or the scalar product to find an angle and then ½ *ab*sin *C* for the area. The problem with some responses was that the points of intersection were not clearly identified and the direction vectors of L_1 and L_2 were used without justification. This lost quite a few marks, as in the following script (Exemplar 4), where the candidate has assumed without justification that the points of intersection are (0, 0, 0), (2, 3, 1) and (1, 2, –4); the correct answer may therefore be regarded as fortuitous.

Exemplar 4

$l_1 : \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$ $\quad l_2 : \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-4\end{pmatrix}$

$l_3 : \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\-4\end{pmatrix} + \sigma\begin{pmatrix}1\\1\\5\end{pmatrix}$



• $d_1 \times d_2 = \begin{vmatrix} i & 2 & 1 \\ j & 3 & 2 \\ k & 1 & -4 \end{vmatrix}$

$= i\begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} - j\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$

$= i(-12-2) - j(-8-1) + k(4-3) = -14i + 9j + k$

• $|d_1 \times d_2| = \sqrt{14^2 + 9^2 + 1^2} = \sqrt{278}$

$\text{Area} = \frac{1}{2}|d_1 \times d_2| = \boxed{\frac{1}{2}\sqrt{278}} \approx \underline{8.34}$

as $|d_1 \times d_2| = |d_1||d_2|\sin\theta\,\hat{n}$

and $\text{Area} = \frac{1}{2}|d_1||d_2|\sin\theta$

Question 13 (a)

13 **(a)** Using the logarithmic form of $\operatorname{arcosh} x$, prove that the derivative of $\operatorname{arcosh} x$ is $\dfrac{1}{\sqrt{x^2-1}}$. **[5]**

Some higher ability candidates differentiated the ln form correctly, but did not see the cancelling to the given answer. Some solutions used the exponential definition of cosh x as a starting point, but these gained no marks.

Question 13 (b)

(b) Hence find $\int_1^2 \operatorname{arcosh} x \, dx$, giving your answer in exact logarithmic form. **[5]**

This is a fairly standard book example of integration by parts and was recognised as such by most of the higher ability candidates. The integration of $x\sqrt{(x^2-1)}$ proved to be a stumbling point for some solutions; it certainly helps if they can recognise this by inspection at this level of complexity.

Question 13 (c)

(c) Ali tries to evaluate $\int_0^1 \operatorname{arcosh} x \, dx$ using his calculator, and gets an 'error'. Explain why. **[1]**

The best explanations referred to the domain of arcosh x here.

Question 14 (a) (i)

14 Three planes have equations

$$\begin{aligned} -x+ay \quad &= 2 \\ 2x+3y+z &= -3 \\ x+by+z &= c \end{aligned}$$

where a, b and c are constants.

(a) In the case where the planes **do not** intersect at a unique point,

(i) find b in terms of a, **[4]**

This was well done. Nearly all candidates equated the determinant to zero, even though there were occasional arithmetic slips. Some candidates responded with a in terms of b.

Question 14 (a) (ii)

(ii) find the value of c for which the planes form a sheaf. **[3]**

Subtracting the second equation from the third and comparing the result with the first gives the answer quite quickly. More generally, the correct approach involves getting 2 equations in one of the variables x, y and z and using $b = a + 3$ to find the value of c that achieves consistency.

Question 14 (b)

(b) In the case where $b = a$ and $c = 1$, find the coordinates of the point of intersection of the planes in terms of a. **[6]**

Many candidates achieved the 3 method marks for an attempt to find the inverse matrix, the determinant and pre-multiplying, however the 'A' marks was a challenge. Some candidates used elimination rather than a matrix approach, but here again accurate work was needed to arrive at the correct solution.

Question 15

15 In this question you must show detailed reasoning.

Show that $\displaystyle\int_{\frac{3}{4}}^{\frac{3}{2}} \frac{1}{\sqrt{4x^2 - 4x + 2}}\,dx = \frac{1}{2}\ln\left(\frac{3+\sqrt{5}}{2}\right)$. **[8]**

This unstructured detailed reasoning question demanded accurate work following the initial recognition of completing the square to pick up all marks.

	AfL	Good candidates often lost the final 2 marks by not showing their rationalising of the denominator of the ln fraction. This was required for a 'Detailed Reasoning' question response.

Question 16 (a)

16 (a) Show that $(2 - e^{i\theta})(2 - e^{-i\theta}) = 5 - 4\cos\theta$. **[3]**

Series C and S are defined by

$$C = \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta + \ldots + \frac{1}{2^n}\cos n\theta,$$

$$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots + \frac{1}{2^n}\sin n\theta.$$

This was quite well done, either using $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ (which may be assumed) or $e^{i\theta} = \cos\theta + i\sin\theta$.

Question 16 (b)

(b) Show that $C = \dfrac{2^n(2\cos\theta - 1) - 2\cos(n+1)\theta + \cos n\theta}{2^n(5 - 4\cos\theta)}$. **[9]**

Most candidates could make a start with this using the $C + \mathrm{i}S$ method. Many then recognised the geometric series and used the sum formula, then used the result of part (a) to get the $5 - 4\cos\theta$ in the denominator. Marks were lost for assuming the series was infinite and omitting or making errors with the 2^n terms; only those candidates that worked accurately gained full marks. The neatness of using the exponential form in examples of this complexity is almost essential to keeping the working concise.

Question 17 (a)

17 A cyclist accelerates from rest for 5 seconds then brakes for 5 seconds, coming to rest at the end of the 10 seconds. The total mass of the cycle and rider is m kg, and at time t seconds, for $0 \leqslant t \leqslant 10$, the cyclist's velocity is $v\,\mathrm{m\,s^{-1}}$.

A resistance to motion, modelled by a force of magnitude $0.1mv$ N, acts on the cyclist during the whole 10 seconds.

(a) Explain why modelling the resistance to motion in this way is likely to be more realistic than assuming this force is constant. **[1]**

Recognising that resistance increased with velocity was common. This mark was gained by many candidates.

Question 17 (b)

During the braking phase of the motion, for $5 \leqslant t \leqslant 10$, the brakes apply an additional constant resistance force of magnitude $2m$ N and the cyclist does not provide any driving force.

(b) Show that, for $5 \leqslant t \leqslant 10$, $\dfrac{\mathrm{d}v}{\mathrm{d}t} + 0.1v = -2$. **[1]**

Some tried to work back from the differential equation, but to gain the mark a clear statement of Newton's 2nd Law (or a = F/m) was required.

Question 17 (c) (i)

(c) **(i)** Solve the differential equation in part **(b)**. **[5]**

All 3 methods given in the mark scheme were seen from candidates. Arguably the easiest was to use the integrating factor and this also proved to be the most efficient. Using separation of variables required candidates to make sure that minuses were cleared before integrating to get ln((2+0.1v) = … and errors here spoiled some solutions. The boundary condition t = 10, v = 0 was perhaps unexpected and caught some out.

Question 17 (c) (ii)

(ii) Hence find the velocity of the cyclist when $t = 5$. **[1]**

Where part (i) was correct, candidates almost always picked up this mark.

Question 17 (d)

During the acceleration phase $(0 \leqslant t \leqslant 5)$, the cyclist applies a driving force of magnitude directly proportional to t.

(d) Show that, for $0 \leqslant t \leqslant 5$, $\frac{\mathrm{d}v}{\mathrm{d}t} + 0.1v = \lambda t$, where λ is a positive constant. **[1]**

Again, a clear statement of F = ma was required. Taking $m\lambda$ as the constant of proportionality helps not to spoil the division by *m*.

Question 17 (e) (i)

(e) **(i)** Show by integration that, for $0 \leqslant t \leqslant 5$, $v = 10\lambda(t - 10 + 10\mathrm{e}^{-0.1t})$. **[5]**

Having the correct equation for *v* helped here. The integrating factor was well done and from there most then used integration by parts and substituted $t = 0$, $v = 0$ to achieve the given answer.

Question 17 (e) (ii)

(ii) Hence find λ. **[2]**

Using the velocity at $t = 5$ to find λ gained a method mark. A correct solution to part *c*(i) was needed to find the correct value of λ.

Question 17 (f)

(f) Find the total distance, to the nearest metre, travelled by the cyclist during the motion. **[6]**

Some candidates showed signs of fatigue in this final question, but quite a few candidates gained at least 3 marks for attempts to integrate the velocity equation for $t = 0$ to 5, and $t = 5$ to 10.

Supporting you

For further details of this qualification please visit the subject webpage.

Review of results

If any of your students' results are not as expected, you may wish to consider one of our review of results services. For full information about the options available visit the OCR website. If university places are at stake you may wish to consider priority service 2 reviews of marking which have an earlier deadline to ensure your reviews are processed in time for university applications.

activeresults

Review students' exam performance with our free online results analysis tool. Available for GCSE, A Level and Cambridge Nationals.

It allows you to:

- review and run analysis reports on exam performance
- analyse results at question and/or topic level*
- compare your centre with OCR national averages
- identify trends across the centre
- facilitate effective planning and delivery of courses
- identify areas of the curriculum where students excel or struggle
- help pinpoint strengths and weaknesses of students and teaching departments.

*To find out which reports are available for a specific subject, please visit ocr.org.uk/administration/support-and-tools/active-results/

Find out more at ocr.org.uk/activeresults

CPD Training

Attend one of our popular CPD courses to hear exam feedback directly from a senior assessor or drop in to an online Q&A session.

Please find details for all our courses on the relevant subject page on our website.

www.ocr.org.uk

The small print

OCR Resources: *the small print*

OCR's resources are provided to support the delivery of OCR qualifications, but in no way constitute an endorsed teaching method that is required by OCR. Whilst every effort is made to ensure the accuracy of the content, OCR cannot be held responsible for any errors or omissions within these resources. We update our resources on a regular basis, so please check the OCR website to ensure you have the most up to date version.

This resource may be freely copied and distributed, as long as the OCR logo and this small print remain intact and OCR is acknowledged as the originator of this work.

Our documents are updated over time. Whilst every effort is made to check all documents, there may be contradictions between published support and the specification, therefore please use the information on the latest specification at all times. Where changes are made to specifications these will be indicated within the document, there will be a new version number indicated, and a summary of the changes. If you do notice a discrepancy between the specification and a resource please contact us at: resources.feedback@ocr.org.uk.

Whether you already offer OCR qualifications, are new to OCR, or are considering switching from your current provider/awarding organisation, you can request more information by completing the Expression of Interest form which can be found here: www.ocr.org.uk/expression-of-interest

Please get in touch if you want to discuss the accessibility of resources we offer to support delivery of our qualifications: resources.feedback@ocr.org.uk

Looking for a resource?

There is now a quick and easy search tool to help find **free** resources for your qualification:

www.ocr.org.uk/i-want-to/find-resources/

www.ocr.org.uk

OCR Customer Support Centre

General qualifications

Telephone 01223 553998
Facsimile 01223 552627
Email general.qualifications@ocr.org.uk

OCR is part of Cambridge Assessment, a department of the University of Cambridge. *For staff training purposes and as part of our quality assurance programme your call may be recorded or monitored.*

Cambridge Assessment

CERTIFIED ISO 9001

UKAS MANAGEMENT SYSTEMS 001