OCR
Oxford Cambridge and RSA

Tuesday 18 January 2022 – Afternoon

Level 3 Cambridge Technical in Engineering

05823/05824/05825/05873 Unit 23: Applied mathematics for engineering

Time allowed: 2 hours
C305/2201

You must have:
- the Formula Booklet for Level 3 Cambridge Technical in Engineering (inside this document)
- a ruler (cm/mm)
- a scientific calculator

Please write clearly in black ink.

Centre number | | | | | | Candidate number | | | | |

First name(s) ______________________________

Last name ______________________________

Date of birth | D | D | M | M | Y | Y | Y | Y |

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided. If you need extra space use the lined pages at the end of this booklet. The question numbers must be clearly shown.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- The acceleration due to gravity is denoted by g m s^{-2}. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.

INFORMATION
- The total mark for this paper is **80**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **20** pages.

ADVICE
- Read each question carefully before you start your answer.

FOR EXAMINER USE ONLY	
Question No	**Mark**
1	/11
2	/11
3	/11
4	/9
5	/14
6	/13
7	/11
Total	**/80**

Answer **all** the questions.

1

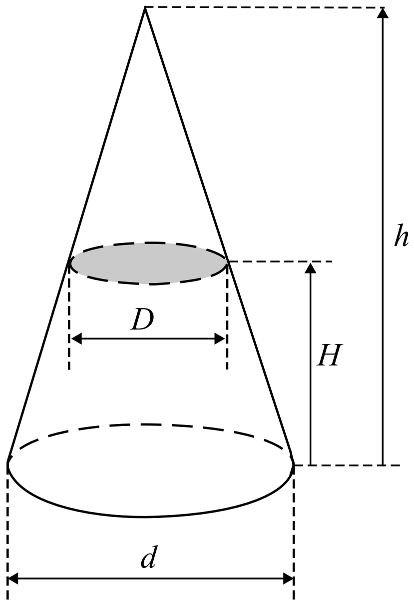

Fig. 1

(a) The cone shown in **Fig. 1** has a circular base with diameter d and height h. At a height H above the base the diameter is D.

Show that $h = \frac{dH}{d-D}$.

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(b)

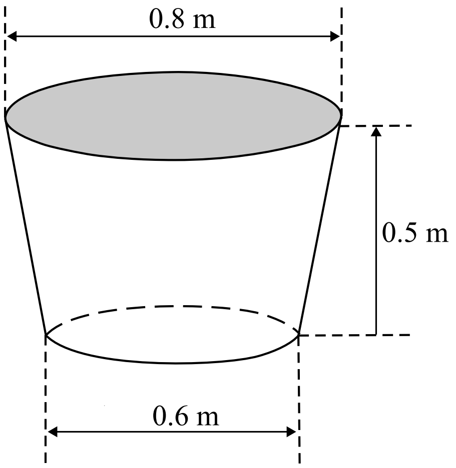

Fig. 2

The circular water tub shown in **Fig. 2** is in the shape of the lower part of a cone like that shown in part **(a)**, but inverted. The base of the tub has diameter 0.6 m; the top has diameter 0.8 m. The height of the tub is 0.5 m.

(i) Calculate the volume of water in the tub when it is completely full.

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(ii) The tub is made from strong plastic.
Calculate the outside surface area of the plastic, including the base.

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2 The relationship between torque, τ N m, and rotational speed, N rpm, for a particular DC motor for speeds between approximately 800 and 8000 rpm is modelled by the equation $N = \frac{a}{\tau} + b\tau + 1000$, where a and b are constants.

(i) Experiments have shown that when $\tau = 20$, $N = 6000$ and when $\tau = 100$, $N = 1500$.

Use this information to calculate the values of a and b. Give your answers correct to 3 decimal places.

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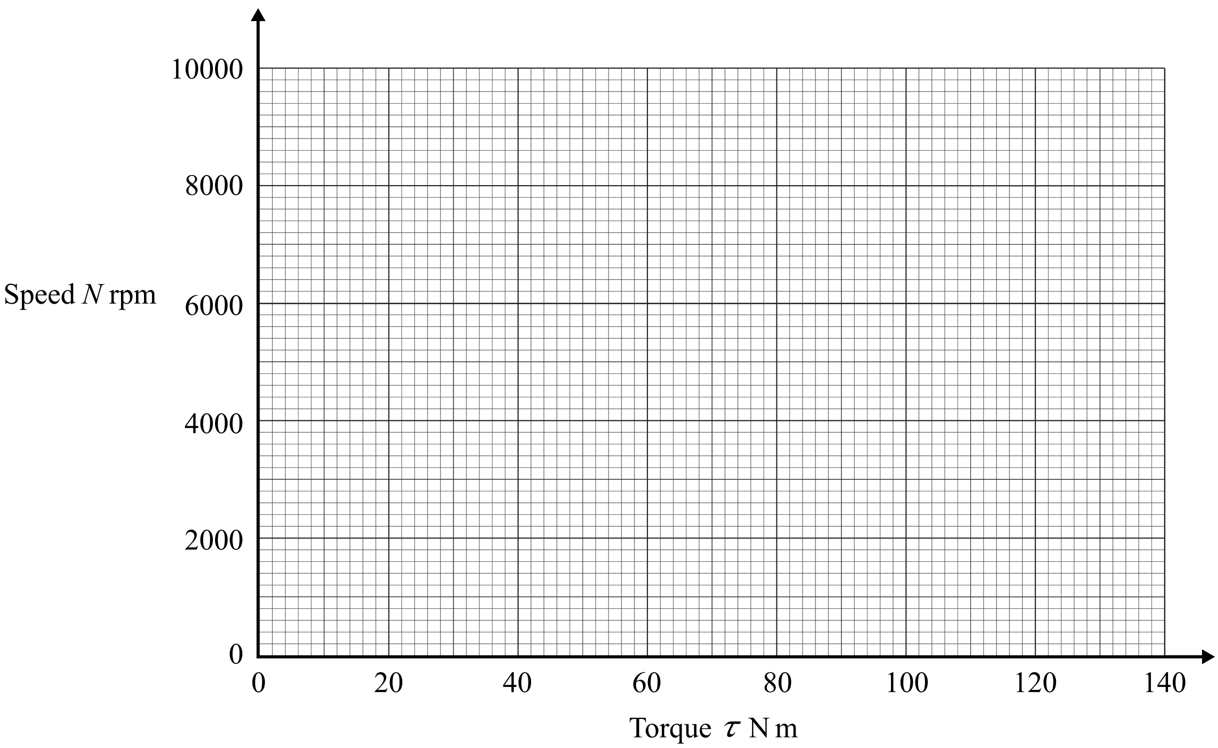
(ii) Sketch a graph of N against τ on the grid below for values of τ between 20 and 100.



[2]

(iii) The power of the motor, P W, is given by $P = \omega\tau$ where ω is the speed of the motor expressed in **radians per second**.

Use calculus to find the value of τ that will produce the maximum power of the motor and calculate this power.

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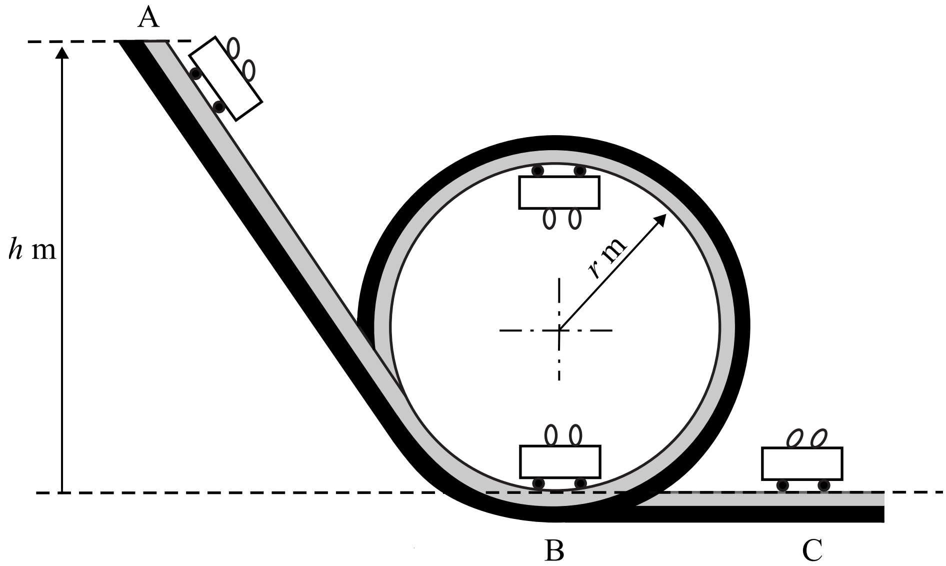

Fig. 3

Fig. 3 shows part of a roller coaster track. Passenger cars start from rest at point A, which is h m higher than the lowest point, B. Cars travel down a uniform slope to point B, where they enter a circular loop of radius r m. After travelling round the loop, cars continue on a horizontal track to point C. The whole track is modelled as being in a single vertical plane.

When a car of mass m kg is travelling round the loop with a speed of v m s^{-1} a force of $\frac{mv^2}{r}$ N acts on the track away from the centre of the loop.

At the bottom of the loop the total downward force acting on the track away from the centre of the loop is $(\frac{mv^2}{r}+mg)$ N, where mg is the component of the total force due to gravity.

At the top of the loop the total upward force acting on the track away from the centre of the loop is $(\frac{mv^2}{r}-mg)$ N. Provided that $\frac{mv^2}{r} \geq mg$ the car will not fall off the track.

Safety rules state that $\frac{mv^2}{r}$ must be at least $1.25mg$ at every point on the roller coaster track.

In this question $r = 8$; you should assume that all frictional forces opposing the motion of the car can be ignored and that the total energy of a car at any point on the track is conserved.

(i) Calculate the minimum speed of a car at the top of the loop so that $\frac{mv^2}{r} \geq 1.25mg$.

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(ii) Using energy considerations, calculate the minimum value of h required to achieve the minimum speed at the top of the loop, as found in part **(i)**.

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(iii) Using the value of h calculated in part **(ii)**, calculate the speed of the car at B.

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(iv) Calculate the total force acting on the track at the bottom of the loop when $m = 1000$.

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(v) If frictional forces are not ignored, how would your answers change and what would be the practical implications of this?

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4 In this question you must express all numerical values exactly, not as decimals. For example, sin 60° should be expressed as $\frac{\sqrt{3}}{2}$. Exact values of common trigonometric functions can be found in section 1.4.1 of the Formula Booklet.

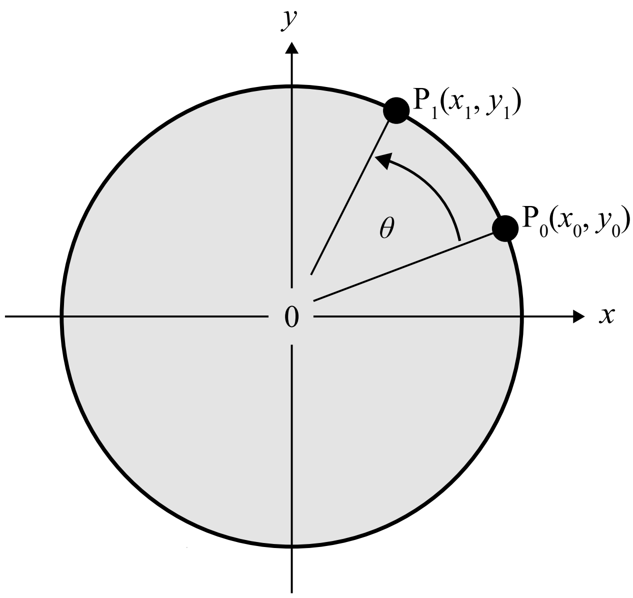

Fig. 4

Fig. 4 shows a flywheel with its centre at the origin of a Cartesian axis system (x, y).
A timing mark on the circumference of the flywheel initially has a position P_0 with coordinates (x_0, y_0). The flywheel is rotated though an angle θ about its centre in an anticlockwise direction after which the timing mark has moved to a new position P_1 with coordinates (x_1, y_1). The coordinates of the timing mark after rotation are given by the following matrix equation.

$$\mathbf{x_1} = \mathbf{A}.\mathbf{x_0}, \quad \text{where } \mathbf{x_0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } \mathbf{x_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

(i) The timing mark is initially at position P_0 with coordinates (2, 0); the flywheel is rotated 30° anticlockwise and the timing mark moves to position P_1.

Write the matrix equation $\mathbf{x_1} = \mathbf{A}.\mathbf{x_0}$ so that it can be used to find the elements of the column vector $\mathbf{x_1}$ representing position P_1. Use exact values for the elements of **A**.

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(ii) Use your answer to part **(i)** to find the exact values of the elements of $\mathbf{x_1}$.

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(iii) The flywheel is now rotated a further 45° anticlockwise about its centre so that the timing mark is at a new position P_2 with coordinates (x_2, y_2) .

Use the matrix equation $\mathbf{x_2 = A.x_1}$ to find the elements of $\mathbf{x_2}$.

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(iv) In a new situation the timing mark is again at position P_0. The flywheel is rotated in an anticlockwise direction through angle θ_a followed by a further rotation in the anticlockwise direction through angle θ_b. The timing mark is then at position P_3, which has column vector $\mathbf{x_3}$, given by the following matrix equation.

$$\mathbf{x_3 = B.x_0} \quad \text{where } \mathbf{B} = \begin{bmatrix} \cos\theta_a & -\sin\theta_a \\ \sin\theta_a & \cos\theta_a \end{bmatrix} . \begin{bmatrix} \cos\theta_b & -\sin\theta_b \\ \sin\theta_b & \cos\theta_b \end{bmatrix}$$

Find the matrix **B** when $\theta_a = 45°$ and $\theta_b = 60°$, giving the elements as exact values.

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5 When a linear dynamic system experiences a sinusoidal input of the form $\sin(\omega t)$ the behaviour of its steady state output has the form $A\sin(\omega t + \alpha)$. The amplitude, A, and the phase angle, α, depend on the physical characteristics of the components used in the system and the value of ω. The output of the system can be characterised by a complex transfer function, $G(\mathrm{j}\omega)$, from which values of A and α can be calculated for particular values of ω.

(i) For a particular dynamic system $G(\mathrm{j}\omega) = \dfrac{1}{(\mathrm{j}\omega)^2 + 2\mathrm{j}\omega + 4}$.

Show that $G(\mathrm{j}\omega) = a + b\mathrm{j}$,

where $a = \dfrac{4-\omega^2}{(4-\omega^2)^2 + 4\omega^2}$ and $b = \dfrac{-2\omega}{(4-\omega^2)^2 + 4\omega^2}$.

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(ii) Given that $A = \sqrt{a^2 + b^2}$, show that $A = \dfrac{1}{\sqrt{\omega^4 - 4\omega^2 + 16}}$.

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(iii) The value of A is maximised when $\frac{\mathrm{d}A}{\mathrm{d}\omega} = 0$ and $\omega > 0$.
Calculate ω when A is a maximum.

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(iv) Given that $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$, calculate α when A is a maximum.

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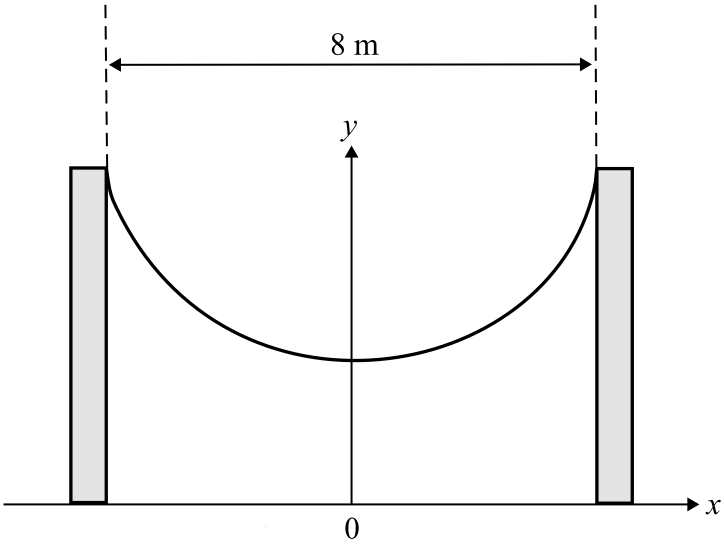

Fig. 5

Two vertical poles of equal height are positioned on horizontal ground 8 metres apart. A cable is suspended between the tops of the two poles as shown in **Fig. 5**. With the point mid-way between the poles at ground level as the origin of a Cartesian axis system (x, y), the cable forms a symmetrical curve with equation $y = e^{x/2} + e^{-x/2} + 1$.

(i) Calculate the x-coordinate of each of the two points on the cable where the height above ground level is 5 metres. You may wish to use the substitution $X = e^{x/2}$.

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(ii) The length of the cable between the two poles is given by $S = 2\int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$.

(*A*) Show that $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\left(e^{x/2} + e^{-x/2}\right)$.

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(*B*) Calculate the length of the cable between the two poles.

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7 Use $g = 10$ in this question for acceleration due to gravity.

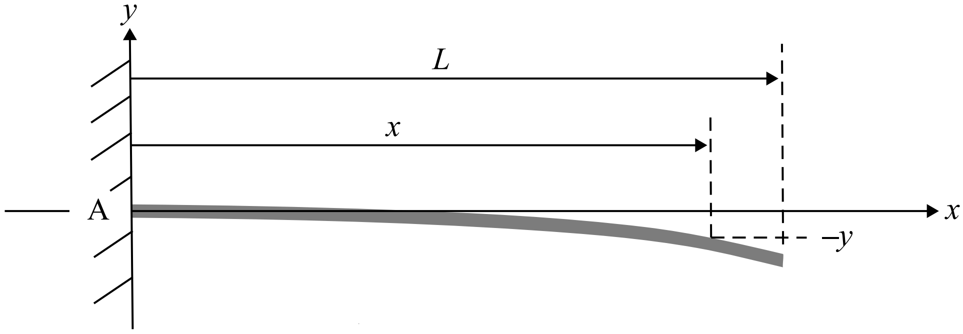

Fig. 6

Fig. 6 shows a uniform steel cantilever beam of length L m fixed at point A. The total weight of the beam is W N. The weight of the beam causes it to be deflected downwards at all points along its length away from point A. By treating point A as the origin of a Cartesian axis system (x, y) the deflection, $-y$ m, and the distance, x m, along the beam from point A, are related by the equation

$$EI\,\frac{d^2y}{dx^2} = -\frac{W(L-x)^2}{2L},$$

where E Pa is Young's modulus for the steel in the beam and I m^4 is the second moment of area about the central horizontal axis of the beam's cross-section (also called the moment of inertia).

(i) By using integration twice and applying the boundary conditions

$y = \frac{dy}{dx} = 0$ when $x = 0$,

show that $y = -\frac{Wx^2(6L^2 - 4Lx + x^2)}{24EIL}$.

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(ii) Show that the maximum deflection at the free end of the beam is given by $y_{\max} = -\frac{WL^3}{8EI}$.

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(iii) A particular uniform steel cantilever beam with an I-shape cross-section has a length of 10 m and a total weight of 1600 N. For this beam $I = 10^{-5}$ and $E = 200 \times 10^9$. Calculate the maximum deflection of this beam.

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(iv) The I-shape cross-section beam is now replaced by a new uniform beam which has a rectangular cross-section measuring a m high and $\frac{a}{4}$ m wide. The new beam is also 10 m in length and is made from steel with a density of 8000 kg m^{-3}.

(*A*) Derive a formula for the total weight of the new beam, giving your answer in terms of a.

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(*B*) For the new beam $I = \frac{a^4}{48}$ and the steel used has $E = 200 \times 10^9$. Find the value of a which will cause the maximum deflection of the beam to be the same as the value calculated for the first beam in part **(iii)**.

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END OF QUESTION PAPER

ADDITIONAL ANSWER SPACE

If additional answer space is required, you should use the following lined pages. The question numbers must be clearly shown – for example, 5(i) or 7(ii).