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A Level Further Mathematics A (H245)

Formulae Booklet

A Level Further Mathematics A

INSTRUCTIONS

- Do **not** send this Booklet for marking. Keep it in the centre or recycle it.

INFORMATION

- This document has **16** pages.

Pure Mathematics

Arithmetic series

$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$S_n = \dfrac{a(1-r^n)}{1-r}$

$S_\infty = \dfrac{a}{1-r}$ for $|r| < 1$

Binomial series

$(a+b)^n = a^n + {}^n\mathrm{C}_1 a^{n-1}b + {}^n\mathrm{C}_2 a^{n-2}b^2 + \ldots + {}^n\mathrm{C}_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$

where ${}^n\mathrm{C}_r = {}_n\mathrm{C}_r = \begin{pmatrix} n \\ r \end{pmatrix} = \dfrac{n!}{r!\,(n-r)!}$

$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dfrac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \quad (|x| < 1,\ n \in \mathbb{R})$

Series

$\sum_{r=1}^{n} r^2 = \dfrac{1}{6}n(n+1)(2n+1), \ \sum_{r=1}^{n} r^3 = \dfrac{1}{4}n^2(n+1)^2$

Maclaurin series

$\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x + \dfrac{\mathrm{f}''(0)}{2!}x^2 + \ldots + \dfrac{\mathrm{f}^{(r)}(0)}{r!}x^r + \ldots$

$\mathrm{e}^x = \exp(x) = 1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^r}{r!} + \ldots$ for all x

$\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots + (-1)^{r+1}\dfrac{x^r}{r} + \ldots (-1 < x \leq 1)$

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots + (-1)^r \dfrac{x^{2r+1}}{(2r+1)!} + \ldots$ for all x

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots + (-1)^r \dfrac{x^{2r}}{(2r)!} + \ldots$ for all x

$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dfrac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \quad (|x| < 1,\ n \in \mathbb{R})$

Matrix transformations

Reflection in the line $y = \pm x$: $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$

Anticlockwise rotation through θ about O: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Rotations through θ about the coordinate axes. The direction of positive rotation is taken to be anticlockwise when looking towards the origin from the positive side of the axis of rotation.

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\arcsin x$ or $\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$ or $\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$ or $\tan^{-1} x$	$\dfrac{1}{1+x^2}$

Quotient rule $y = \dfrac{u}{v}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Integration

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$$

$$\int \mathrm{f}'(x)(\mathrm{f}(x))^n\,\mathrm{d}x = \frac{1}{n+1}(\mathrm{f}(x))^{n+1} + c$$

Integration by parts $\displaystyle\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

The mean value of $\mathrm{f}(x)$ on the interval $[a, b]$ is $\frac{1}{b-a}\int_a^b \mathrm{f}(x)\,\mathrm{d}x$

Area of sector enclosed by polar curve is $\frac{1}{2}\int r^2\,\mathrm{d}\theta$

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$ or $\ln(x+\sqrt{x^2+a^2})$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$ or $\ln(x+\sqrt{x^2-a^2}) \quad (x > a)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x)=0: x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Complex numbers

Circles: $\lvert z-a\rvert = k$

Half lines: $\arg(z-a) = \alpha$

Lines: $\lvert z-a\rvert = \lvert z-b\rvert$

De Moivre's theorem: $\{r(\cos\theta+\mathrm{i}\sin\theta)\}^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$

Roots of unity: The roots of $z^n = 1$ are given by $z = \exp\left(\dfrac{2\pi k}{n}\mathrm{i}\right)$ for $k = 0, 1, 2, \ldots, n-1$

Vectors and 3-D coordinate geometry

Cartesian equation of the line through the point A with position vector $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ in direction $\mathbf{u} = u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$ is $\dfrac{x-a_1}{u_1} = \dfrac{y-a_2}{u_2} = \dfrac{z-a_3}{u_3} (= \lambda)$

Cartesian equation of a plane $n_1x+n_2y+n_3z+d = 0$

Vector product: $\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & a_1 & b_1 \\ \mathbf{j} & a_2 & b_2 \\ \mathbf{k} & a_3 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$

The distance between skew lines is $D = \frac{|(\mathbf{b}-\mathbf{a}).\mathbf{n}|}{|\mathbf{n}|}$, where **a** and **b** are position vectors of points on each line and **n** is a mutual perpendicular to both lines

The distance between a point and a line is $D = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2+b^2}}$, where the coordinates of the point are (x_1, y_1) and the equation of the line is given by $ax + by = c$

The distance between a point and a plane is $D = \frac{|\mathbf{b}.\mathbf{n} - p|}{|\mathbf{n}|}$, where **b** is the position vector of the point and the equation of the plane is given by $\mathbf{r}.\mathbf{n} = p$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is small and measured in radians

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$$

Hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh^{-1} x = \ln[x + \sqrt{(x^2+1)}]$$

$$\cosh^{-1} x = \ln[x + \sqrt{(x^2-1)}],\ x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ -1 < x < 1$$

Simple harmonic motion

$$x = A\cos(\omega t) + B\sin(\omega t)$$

$$x = R\sin(\omega t + \varphi)$$

Statistics

Probability

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B|A) = \mathrm{P}(B)\mathrm{P}(A|B)$ **or** $\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Standard deviation

$$\sqrt{\frac{\Sigma(x-\overline{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \overline{x}^2} \text{ or } \sqrt{\frac{\Sigma f(x-\overline{x})^2}{\Sigma f}} = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \overline{x}^2}$$

Sampling distributions

For any variable X, $\mathrm{E}(\overline{X}) = \mu$, $\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$ and $\overline{X}$ is approximately normally distributed when n is large enough (approximately $n > 25$)

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ and $\frac{\overline{X}-\mu}{\sigma / \sqrt{n}} \sim \mathrm{N}(0,1)$

Unbiased estimates of the population mean and variance are given by $\frac{\Sigma x}{n}$ and $\frac{n}{n-1}\left(\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2\right)$

Expectation algebra

Use the following results, including the cases where $a = b = \pm 1$ and/or $c = 0$:

1. $\mathrm{E}(aX + bY + c) = a\mathrm{E}(X) + b\mathrm{E}(Y) + c$,

2. if X and Y are independent then $\mathrm{Var}(aX + bY + c) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y)$.

Discrete distributions

X is a random variable taking values x_i in a discrete distribution with $\mathrm{P}(X = x_i) = p_i$

Expectation: $\mu = \mathrm{E}(X) = \sum x_i p_i$

Variance: $\sigma^2 = \mathrm{Var}(X) = \sum (x_i - \mu)^2 p_i = \sum x_i^2 p_i - \mu^2$

	$P(X=x)$	$\mathrm{E}(X)$	$\mathrm{Var}(X)$
Binomial $\mathrm{B}(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Uniform distribution over 1, 2, …, n, $\mathrm{U}(n)$	$\frac{1}{n}$	$\frac{n+1}{2}$	$\frac{1}{12}(n^2-1)$
Geometric distribution $\mathrm{Geo}(p)$	$(1-p)^{x-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson $\mathrm{Po}(\lambda)$	$e^{-\lambda}\frac{\lambda^x}{x!}$	λ	λ

Continuous distributions

X is a continuous random variable with probability density function (p.d.f.) $\mathrm{f}(x)$

Expectation: $\mu = \mathrm{E}(X) = \int x\,\mathrm{f}(x)\,\mathrm{d}x$

Variance: $\sigma^2 = \mathrm{Var}(X) = \int (x-\mu)^2\,\mathrm{f}(x)\,\mathrm{d}x = \int x^2\,\mathrm{f}(x)\,\mathrm{d}x - \mu^2$

Cumulative distribution function $\mathrm{F}(x) = \mathrm{P}(X \le x) = \int_{-\infty}^{x} \mathrm{f}(t)\,\mathrm{d}t$

	p.d.f.	$\mathrm{E}(X)$	$\mathrm{Var}(X)$
Continuous uniform distribution over $[a, b]$	$\frac{1}{b-a}$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$
Exponential	$\lambda \mathrm{e}^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal $N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $P(Z \le z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Non-parametric tests

Goodness-of-fit test and contingency tables: $\sum \frac{(O_i - E_i)^2}{E_i} \sim \chi^2_v$

Approximate distributions for large samples

Wilcoxon Signed Rank test: $T \sim \mathrm{N}\left(\frac{1}{4}n(n+1), \frac{1}{24}n(n+1)(2n+1)\right)$

Wilcoxon Rank Sum test (samples of sizes m and n, with $m \le n$) :

$$W \sim \mathrm{N}\left(\tfrac{1}{2}m(m+n+1), \tfrac{1}{12}mn(m+n+1)\right)$$

Correlation and regression

For a sample of n pairs of observations (x_i, y_i)

$$S_{xx} = \sum (x_i - \overline{x})^2 = \sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}, \quad S_{yy} = \sum (y_i - \overline{y})^2 = \sum y_i^2 - \frac{\left(\sum y_i\right)^2}{n},$$

$$S_{xy} = \sum (x_i - \overline{x})(y_i - \overline{y}) = \sum x_i y_i - \frac{\sum x_i \sum y_i}{n}$$

Product moment correlation coefficient: $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = \frac{\sum x_i y_i - \frac{\sum x_i \sum y_i}{n}}{\sqrt{\left[\left(\sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}\right)\left(\sum y_i^2 - \frac{\left(\sum y_i\right)^2}{n}\right)\right]}}$

The regression coefficient of y on x is $b = \frac{S_{xy}}{S_{xx}} = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{\sum (x_i - \overline{x})^2}$

Least squares regression line of y on x is $y = a + bx$ where $a = \overline{y} - b\overline{x}$

Spearman's rank correlation coefficient: $r_s = 1 - \frac{6\sum d_i^2}{n(n^2 - 1)}$

Critical values for the product moment correlation coefficient, r

1-Tail Test	5%	2½%	1%	½%
2-Tail Test	10%	5%	2%	1%
n				
1	-	-	-	-
2	-	-	-	-
3	0.9877	0.9969	0.9995	0.9999
4	0.9000	0.9500	0.9800	0.9900
5	0.8054	0.8783	0.9343	0.9587
6	0.7293	0.8114	0.8822	0.9172
7	0.6694	0.7545	0.8329	0.8745
8	0.6215	0.7067	0.7887	0.8343
9	0.5822	0.6664	0.7498	0.7977
10	0.5494	0.6319	0.7155	0.7646
11	0.5214	0.6021	0.6851	0.7348
12	0.4973	0.5760	0.6581	0.7079
13	0.4762	0.5529	0.6339	0.6835
14	0.4575	0.5324	0.6120	0.6614
15	0.4409	0.5140	0.5923	0.6411
16	0.4259	0.4973	0.5742	0.6226
17	0.4124	0.4821	0.5577	0.6055
18	0.4000	0.4683	0.5425	0.5897
19	0.3887	0.4555	0.5285	0.5751
20	0.3783	0.4438	0.5155	0.5614
21	0.3687	0.4329	0.5034	0.5487
22	0.3598	0.4227	0.4921	0.5368
23	0.3515	0.4132	0.4815	0.5256
24	0.3438	0.4044	0.4716	0.5151
25	0.3365	0.3961	0.4622	0.5052
26	0.3297	0.3882	0.4534	0.4958
27	0.3233	0.3809	0.4451	0.4869
28	0.3172	0.3739	0.4372	0.4785
29	0.3115	0.3673	0.4297	0.4705
30	0.3061	0.3610	0.4226	0.4629
31	0.3009	0.3550	0.4158	0.4556
32	0.2960	0.3494	0.4093	0.4487
33	0.2913	0.3440	0.4032	0.4421
34	0.2869	0.3388	0.3972	0.4357
35	0.2826	0.3338	0.3916	0.4296
36	0.2785	0.3291	0.3862	0.4238
37	0.2746	0.3246	0.3810	0.4182
38	0.2709	0.3202	0.3760	0.4128
39	0.2673	0.3160	0.3712	0.4076
40	0.2638	0.3120	0.3665	0.4026
41	0.2605	0.3081	0.3621	0.3978
42	0.2573	0.3044	0.3578	0.3932
43	0.2542	0.3008	0.3536	0.3887
44	0.2512	0.2973	0.3496	0.3843
45	0.2483	0.2940	0.3457	0.3801
46	0.2455	0.2907	0.3420	0.3761
47	0.2429	0.2876	0.3384	0.3721
48	0.2403	0.2845	0.3348	0.3683
49	0.2377	0.2816	0.3314	0.3646
50	0.2353	0.2787	0.3281	0.3610
51	0.2329	0.2759	0.3249	0.3575
52	0.2306	0.2732	0.3218	0.3542
53	0.2284	0.2706	0.3188	0.3509
54	0.2262	0.2681	0.3158	0.3477
55	0.2241	0.2656	0.3129	0.3445
56	0.2221	0.2632	0.3102	0.3415
57	0.2201	0.2609	0.3074	0.3385
58	0.2181	0.2586	0.3048	0.3357
59	0.2162	0.2564	0.3022	0.3328
60	0.2144	0.2542	0.2997	0.3301

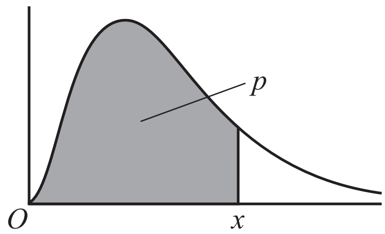
Critical values for Spearman's rank correlation coefficient, r_s

1-Tail Test	5%	2½%	1%	½%
2-Tail Test	10%	5%	2%	1%
n				
1	-	-	-	-
2	-	-	-	-
3	-	-	-	-
4	1.0000	-	-	-
5	0.9000	1.0000	1.0000	-
6	0.8286	0.8857	0.9429	1.0000
7	0.7143	0.7857	0.8929	0.9286
8	0.6429	0.7381	0.8333	0.8810
9	0.6000	0.7000	0.7833	0.8333
10	0.5636	0.6485	0.7455	0.7939
11	0.5364	0.6182	0.7091	0.7545
12	0.5035	0.5874	0.6783	0.7273
13	0.4835	0.5604	0.6484	0.7033
14	0.4637	0.5385	0.6264	0.6791
15	0.4464	0.5214	0.6036	0.6536
16	0.4294	0.5029	0.5824	0.6353
17	0.4142	0.4877	0.5662	0.6176
18	0.4014	0.4716	0.5501	0.5996
19	0.3912	0.4596	0.5351	0.5842
20	0.3805	0.4466	0.5218	0.5699
21	0.3701	0.4364	0.5091	0.5558
22	0.3608	0.4252	0.4975	0.5438
23	0.3528	0.4160	0.4862	0.5316
24	0.3443	0.4070	0.4757	0.5209
25	0.3369	0.3977	0.4662	0.5108
26	0.3306	0.3901	0.4571	0.5009
27	0.3242	0.3828	0.4487	0.4915
28	0.3180	0.3755	0.4401	0.4828
29	0.3118	0.3685	0.4325	0.4749
30	0.3063	0.3624	0.4251	0.4670
31	0.3012	0.3560	0.4185	0.4593
32	0.2962	0.3504	0.4117	0.4523
33	0.2914	0.3449	0.4054	0.4455
34	0.2871	0.3396	0.3995	0.4390
35	0.2829	0.3347	0.3936	0.4328
36	0.2788	0.3300	0.3882	0.4268
37	0.2748	0.3253	0.3829	0.4211
38	0.2710	0.3209	0.3778	0.4155
39	0.2674	0.3168	0.3729	0.4103
40	0.2640	0.3128	0.3681	0.4051
41	0.2606	0.3087	0.3636	0.4002
42	0.2574	0.3051	0.3594	0.3955
43	0.2543	0.3014	0.3550	0.3908
44	0.2513	0.2978	0.3511	0.3865
45	0.2484	0.2945	0.3470	0.3822
46	0.2456	0.2913	0.3433	0.3781
47	0.2429	0.2880	0.3396	0.3741
48	0.2403	0.2850	0.3361	0.3702
49	0.2378	0.2820	0.3326	0.3664
50	0.2353	0.2791	0.3293	0.3628
51	0.2329	0.2764	0.3260	0.3592
52	0.2307	0.2736	0.3228	0.3558
53	0.2284	0.2710	0.3198	0.3524
54	0.2262	0.2685	0.3168	0.3492
55	0.2242	0.2659	0.3139	0.3460
56	0.2221	0.2636	0.3111	0.3429
57	0.2201	0.2612	0.3083	0.3400
58	0.2181	0.2589	0.3057	0.3370
59	0.2162	0.2567	0.3030	0.3342
60	0.2144	0.2545	0.3005	0.3314

Critical values for the χ^2 distribution

If X has a χ^2 distribution with v degrees of freedom then, for each pair of values of p and v, the table gives the value of x such that

$$P(X \le x) = p.$$



p	0.01	0.025	0.05	0.90	0.95	0.975	0.99	0.995	0.999
$v = 1$	$0.0^3 1571$	$0.0^3 9821$	$0.0^2 3932$	2.706	3.841	5.024	6.635	7.879	10.83
2	0.02010	0.05064	0.1026	4.605	5.991	7.378	9.210	10.60	13.82
3	0.1148	0.2158	0.3518	6.251	7.815	9.348	11.34	12.84	16.27
4	0.2971	0.4844	0.7107	7.779	9.488	11.14	13.28	14.86	18.47
5	0.5543	0.8312	1.145	9.236	11.07	12.83	15.09	16.75	20.51
6	0.8721	1.237	1.635	10.64	12.59	14.45	16.81	18.55	22.46
7	1.239	1.690	2.167	12.02	14.07	16.01	18.48	20.28	24.32
8	1.647	2.180	2.733	13.36	15.51	17.53	20.09	21.95	26.12
9	2.088	2.700	3.325	14.68	16.92	19.02	21.67	23.59	27.88
10	2.558	3.247	3.940	15.99	18.31	20.48	23.21	25.19	29.59
11	3.053	3.816	4.575	17.28	19.68	21.92	24.73	26.76	31.26
12	3.571	4.404	5.226	18.55	21.03	23.34	26.22	28.30	32.91
13	4.107	5.009	5.892	19.81	22.36	24.74	27.69	29.82	34.53
14	4.660	5.629	6.571	21.06	23.68	26.12	29.14	31.32	36.12
15	5.229	6.262	7.261	22.31	25.00	27.49	30.58	32.80	37.70
16	5.812	6.908	7.962	23.54	26.30	28.85	32.00	34.27	39.25
17	6.408	7.564	8.672	24.77	27.59	30.19	33.41	35.72	40.79
18	7.015	8.231	9.390	25.99	28.87	31.53	34.81	37.16	42.31
19	7.633	8.907	10.12	27.20	30.14	32.85	36.19	38.58	43.82
20	8.260	9.591	10.85	28.41	31.41	34.17	37.57	40.00	45.31
21	8.897	10.28	11.59	29.62	32.67	35.48	38.93	41.40	46.80
22	9.542	10.98	12.34	30.81	33.92	36.78	40.29	42.80	48.27
23	10.20	11.69	13.09	32.01	35.17	38.08	41.64	44.18	49.73
24	10.86	12.40	13.85	33.20	36.42	39.36	42.98	45.56	51.18
25	11.52	13.12	14.61	34.38	37.65	40.65	44.31	46.93	52.62
30	14.95	16.79	18.49	40.26	43.77	46.98	50.89	53.67	59.70
40	22.16	24.43	26.51	51.81	55.76	59.34	63.69	66.77	73.40
50	29.71	32.36	34.76	63.17	67.50	71.42	76.15	79.49	86.66
60	37.48	40.48	43.19	74.40	79.08	83.30	88.38	91.95	99.61
70	45.44	48.76	51.74	85.53	90.53	95.02	100.4	104.2	112.3
80	53.54	57.15	60.39	96.58	101.9	106.6	112.3	116.3	124.8
90	61.75	65.65	69.13	107.6	113.1	118.1	124.1	128.3	137.2
100	70.06	74.22	77.93	118.5	124.3	129.6	135.8	140.2	149.4

Wilcoxon signed rank test

W_+ is the sum of the ranks corresponding to the positive differences,

W_- is the sum of the ranks corresponding to the negative differences,

T is the smaller of W_+ and W_-.

For each value of n the table gives the **largest** value of T which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of *T*

	Level of significance			
One Tail	0.05	0.025	0.01	0.005
Two Tail	0.10	0.05	0.02	0.01
$n = 6$	2	0		
7	3	2	0	
8	5	3	1	0
9	8	5	3	1
10	10	8	5	3
11	13	10	7	5
12	17	13	9	7
13	21	17	12	9
14	25	21	15	12
15	30	25	19	15
16	35	29	23	19
17	41	34	27	23
18	47	40	32	27
19	53	46	37	32
20	60	52	43	37

For larger values of n, each of W_+ and W_- can be approximated by the normal distribution with mean $\frac{1}{4}n(n+1)$ and variance $\frac{1}{24}n(n+1)(2n+1)$.

Wilcoxon rank sum test

The two samples have sizes m and n, where $m \le n$.
R_m is the sum of the ranks of the items in the sample of size m.
W is the smaller of R_m and $m(m+n+1)-R_m$.

For each pair of values of m and n, the table gives the **largest** value of W which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of *W*

	Level of significance											
One Tail	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two Tail	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	$m=3$			$m=4$			$m=5$			$m=6$		
3	6	-	-									
4	6	-	-	11	10	-						
5	7	6	-	12	11	10	19	17	16			
6	8	7	-	13	12	11	20	18	17	28	26	24
7	8	7	6	14	13	11	21	20	18	29	27	25
8	9	8	6	15	14	12	23	21	19	31	29	27
9	10	8	7	16	14	13	24	22	20	33	31	28
10	10	9	7	17	15	13	26	23	21	35	32	29

	Level of significance											
One Tail	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
Two Tail	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02	0.1	0.05	0.02
n	$m=7$			$m=8$			$m=9$			$m=10$		
7	39	36	34									
8	41	38	35	51	49	45						
9	43	40	37	54	51	47	66	62	59			
10	45	42	39	56	53	49	69	65	61	82	78	74

For larger values of m and n, the normal distribution with mean $\frac{1}{2}m(m+n+1)$ and variance $\frac{1}{12}mn(m+n+1)$ should be used as an approximation to the distribution of R_m.

Mechanics

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u+v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u}+\mathbf{v})t$

$\mathbf{v}.\mathbf{v} = \mathbf{u}.\mathbf{u} + 2\mathbf{a}.\mathbf{s}$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Newton's experimental law

Between two smooth spheres $v_1 - v_2 = -e(u_1 - u_2)$

Between a smooth sphere with a fixed plane surface $v = -eu$

Motion in a circle

Tangential velocity is $v = r\dot{\theta}$

Radial acceleration is $\frac{v^2}{r}$ or $r\dot{\theta}^2$ towards the centre

Tangential acceleration is $\dot{v} = r\ddot{\theta}$

Centres of mass

Triangular lamina: $\frac{2}{3}$ along median from vertex

Solid hemisphere, radius r: $\frac{3}{8}r$ from centre

Hemispherical shell, radius r: $\frac{1}{2}r$ from centre

Circular arc, radius r, angle at centre 2α: $\frac{r\sin\alpha}{\alpha}$ from centre

Sector of circle, radius r, angle at centre 2α: $\frac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height h: $\frac{1}{4}h$ above the base on the line from centre of base to vertex

Conical shell of height h: $\frac{1}{3}h$ above the base on the line from centre of base to vertex

Discrete Mathematics

Inclusion-exclusion principle
For sets A, B and C:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

The hierarchy of orders

$$\mathrm{O}(1) \subset \mathrm{O}(\log n) \subset \mathrm{O}(n) \subset \mathrm{O}(n \log n) \subset \mathrm{O}(n^2) \subset \mathrm{O}(n^3) \subset \ldots \subset \mathrm{O}(a^n) \subset \mathrm{O}(n!)$$

Sorting algorithms

Bubble sort:

Start at the left hand end of the list unless specified otherwise.

Compare the first and second values and swap if necessary. Then compare the (new) second value with the third value and swap if necessary. Continue in this way until all values have been considered.

Fix the last value then repeat with the reduced list until either there is a pass in which no swaps occur or the list is reduced to length 1, then stop.

Shuttle sort:

Start at the left hand end of the list unless specified otherwise.

Compare the second value with the first and swap if necessary, this completes the first pass. Next compare the third value with the second and swap if necessary, if a swap happened shuttle back to compare the (new) second with the first as in the first pass, this completes the second pass.

Next compare the fourth value with the third and swap if necessary, if a swap happened shuttle back to compare the (new) third value with the second as in the second pass (so if a swap happens shuttle back again). Continue in this way for $n-1$ passes, where n is the length of the list.

Quick sort:

The first value in any sublist will be the pivot, unless specified otherwise.

Working from left to right, write down each value that is smaller than the pivot, then the pivot, then work along the list and write down each value that is not smaller than the pivot. This produces two sublists (one of which may be empty) with the pivot between them and completes the pass.

Next apply this procedure to each of the sublists from the previous pass, unless they consist of a single entry, to produce further sublists. Continue in this way until no sublist has more than one entry.

Network algorithms

Dijkstra's algorithm

START with a graph *G*. At each vertex draw a box, the lower area for temporary labels, the upper left hand area for the order of becoming permanent and the upper right hand area for the permanent label.

STEP 1 Make the given start vertex permanent by giving it permanent label 0 and order label 1.

STEP 2 For each vertex that is not permanent and is connected by an arc to the vertex that has just been made permanent (with permanent label = P), add the arc weight to P. If this is smaller than the best temporary label at the vertex, write this value as the new best temporary label.

STEP 3 Choose the vertex that is not yet permanent which has the smallest best temporary label. If there is more than one such vertex, choose any one of them. Make this vertex permanent and assign it the next order label.

STEP 4 If every vertex is now permanent, or if the target vertex is permanent, use 'trace back' to find the routes or route, then STOP; otherwise return to STEP 2.

Prim's algorithm (graphical version)

START with an arbitrary vertex of *G*.

STEP 1 Add an edge of minimum weight joining a vertex already included to a vertex not already included.

STEP 2 If a spanning tree is obtained STOP; otherwise return to STEP 1.

Prim's algorithm (tabular version)

START with a table (or matrix) of weights for a connected weighted graph.

STEP 1 Cross through the entries in an arbitrary row, and mark the corresponding column.

STEP 2 Choose a minimum entry from the uncircled entries in the marked column(s).

STEP 3 If no such entry exists STOP; otherwise go to STEP 4.

STEP 4 Circle the weight w_{ij} found in STEP 2; mark column i; cross through row i.

STEP 5 Return to STEP 2.

Kruskal's algorithm

START with all the vertices of *G*, but no edges; list the edges in increasing order of weight.

STEP 1 Add an edge of *G* of minimum weight in such a way that no cycles are created.

STEP 2 If a spanning tree is obtained STOP; otherwise return to STEP 1.

Nearest neighbour method

START at a given vertex of G.

STEP 1 Find the least weight arc from this vertex to a vertex that has not already been included (or back to the start vertex if every vertex has been included).

STEP 2 If no such arc exists then the method has stalled STOP; otherwise add this arc to the path.

STEP 3 If a cycle has been found STOP; otherwise return to STEP 1.

Lower bound for travelling salesperson problem

START with all vertices and arcs of G.

STEP 1 Remove a given vertex and all arcs that are directly connected to that vertex, find a minimum spanning tree for the resulting reduced network.

STEP 2 Add the weight of this minimum connector to the sum of the two least weight arcs that had been deleted. This gives a lower bound.

Route inspection problem

START with a list of the odd degree vertices.

STEP 1 For each pair of odd nodes, find the connecting path of least weight.

STEP 2 Group the odd nodes so that the sum of weights of the connecting paths is minimised.

STEP 3 Add this sum to the total weight of the graph STOP.

The simplex algorithm

START with a tableau in standard format.

STEP 1 Choose a column with a negative entry in the objective row (or zero in degenerate cases).

STEP 2 The pivot row is the one for which non-negative value of the entry in the final column divided by the positive value of the entry in the pivot column is minimised. The pivot element is the entry of the pivot row in the chosen column.

STEP 3 Divide all entries in the pivot row by the value of the pivot element.

STEP 4 Add to, or subtract from, all other old rows a multiple of the new pivot row, so that the pivot column ends up consisting of zeroes and a single one, and corresponds to the new basic variable.

STEP 5 If the objective row has no negative entries STOP; otherwise return to STEP 1.

Additional Pure Mathematics

Vector product

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$, where $\mathbf{a}$, $\mathbf{b}$, $\hat{\mathbf{n}}$, in that order, form a right-handed triple.

Surfaces

For 3-D surfaces given in the form $z = \mathrm{f}(x, y)$, the Hessian Matrix is given by $\mathbf{H} = \begin{pmatrix} \mathrm{f}_{xx} & \mathrm{f}_{xy} \\ \mathrm{f}_{yx} & \mathrm{f}_{yy} \end{pmatrix}$.

At a stationary point of the surface:

1. if $|\mathbf{H}| > 0$ and $\mathrm{f}_{xx} > 0$, there is a (local) minimum;
2. if $|\mathbf{H}| > 0$ and $\mathrm{f}_{xx} < 0$, there is a (local) maximum;
3. if $|\mathbf{H}| < 0$ there is a saddle-point;
4. if $|\mathbf{H}| = 0$ then the nature of the stationary point cannot be determined by this test.

The equation of a tangent plane to the curve at a given point $(x, y, z) = (a, b, \mathrm{f}(a, b))$ is

$$z = \mathrm{f}(a, b) + (x - a)\mathrm{f}_x(a, b) + (y - b)\mathrm{f}_y(a, b).$$

Calculus

Arc length $\quad s = \displaystyle\int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$

$$s = \int_a^b \sqrt{(\dot{x}^2 + \dot{y}^2)}\,\mathrm{d}t$$

Surface area of revolution

$$S_x = 2\pi \int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \qquad S_y = 2\pi \int_c^d x\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

$$S_x = 2\pi \int_a^b y(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \qquad S_y = 2\pi \int_c^d x(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$