OCR
Oxford Cambridge and RSA

Friday 21 June 2024 – Afternoon

A Level Further Mathematics A

Y545/01 Additional Pure Mathematics

Time allowed: 1 hour 30 minutes

1422481346

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for A Level Further Mathematics A
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **4** pages.

ADVICE
- Read each question carefully before you start your answer.

1 **(a)** The number N has the base-10 form $N = abba\ abba\ \ldots\ abba$, consisting of blocks of four digits, as shown, where a and b are integers such that $1 \leqslant a < 10$ and $0 \leqslant b < 10$.

Use a standard divisibility test to show that N is always divisible by 11. **[3]**

(b) The number M has the base-n form $M = cddc\ cddc\ \ldots\ cddc$, where $n > 11$ and c and d are integers such that $1 \leqslant c < n$ and $0 \leqslant d < n$.

Show that M is always divisible by a number of the form $k_1 n + k_2$, where k_1 and k_2 are integers to be determined. **[3]**

2 A surface S has equation $z = 4x\sqrt{y} - y\sqrt{x} + y^2$ for $x,\ y \geqslant 0$.

Determine the equation of the tangent plane to S at the point (1, 4, 20). Give your answer in the form $ax + by + cz = d$ where a, b, c and d are integers. **[5]**

3 Determine all integers x for which $x \equiv 1 \pmod{7}$ and $x \equiv 22 \pmod{37}$ and $x \equiv 7 \pmod{67}$. Give your answer in the form $x = qn + r$ for integers n, q, r with $q > 0$ and $0 \leqslant r < q$. **[6]**

4 The vectors **a** and **b** are given by $\mathbf{a} = \begin{pmatrix} p-1 \\ q+2 \\ 2r-3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2p+4 \\ 2q-5 \\ r+3 \end{pmatrix}$, where p, q and r are real numbers.

(a) Given that **b** is **not** a multiple of **a** and that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, determine all possible sets of values of p, q and r. **[3]**

(b) You are given instead that $\mathbf{b} = \lambda\mathbf{a}$, where λ is an integer with $|\lambda| > 1$.

By writing each of p, q and r in terms of λ, show that there is a unique value of λ for which p, q and r are all integers, stating this set of values of p, q and r. **[7]**

5 In a conservation project in a nature reserve, scientists are modelling the population of one species of animal.

The initial population of the species, P_0, is 10 000. After n years, the population is P_n. The scientists believe that the year-on-year change in the population can be modelled by a recurrence relation of the form
$P_{n+1} = 2P_n(1 - kP_n)$ for $n \geqslant 0$, where k is a constant.

(a) The initial aim of the project is to ensure that the population remains constant.

Show that this happens, according to this model, when $k = 0.000\,05$. **[2]**

(b) After a few years, with the population still at 10 000, the scientists suggest increasing the population. One way of achieving this is by adding 50 more of these animals into the nature reserve at the end of each year.

In this scenario, the recurrence system modelling the population (using $k = 0.000\,05$) is given by
$P_0 = 10\,000$ and $P_{n+1} = 2P_n(1 - 0.000\,05P_n) + 50$ for $n \geqslant 0$.

Use your calculator to find the long-term behaviour of P_n predicted by this recurrence system. **[1]**

(c) However, the scientists decide **not** to add any animals at the end of each year. Also, further research predicts that certain factors will remove 2400 animals from the population each year.

(i) Write down a modified form of the recurrence relation given in part **(b)**, that will model the population of these animals in the nature reserve when 2400 animals are removed each year and no additional animals are added. **[1]**

(ii) Use your calculator to find the behaviour of P_n predicted by this modified form of the recurrence relation over the course of the next ten years. **[1]**

(iii) Show algebraically that this modified form of the recurrence relation also gives a constant value of P_n in the long term, which should be stated. **[3]**

(iv) Determine what constant value should replace 0.000 05 in this modified form of the recurrence relation to ensure that the value of P_n remains constant at 10 000. **[2]**

6 The surface C is given by the equation $z = x^2 + y^3 + axy$ for all real x and y, where a is a non-zero real number.

(a) Show that C has two stationary points, one of which is at the origin, and give the coordinates of the second in terms of a. **[6]**

(b) Determine the nature of these stationary points of C. **[5]**

(c) Explain what can be said about the location and nature of the stationary point(s) of the surface given by the equation $z = x^2 + y^3$ for all real x and y. **[2]**

7 Let $I_n = \int_0^2 \frac{x^n}{\sqrt{x^3+1}}\,\mathrm{d}x$ for integers $n > 0$.

(a) By considering the derivative of $\sqrt{x^3+1}$ with respect to x, determine the exact value of I_2. **[2]**

(b) Given that $n > 3$, show that $(2n-1)I_n = 3\times 2^{n-1} - 2(n-2)I_{n-3}$. **[5]**

(c) Hence determine the exact value of $\int_0^2 x^5\sqrt{x^3+1}\,\mathrm{d}x$. **[3]**

8 The group G is cyclic and of order 12.

(a) (i) State the possible orders of all the proper subgroups of G. You must justify your answers. **[2]**

(ii) List all the elements of each of these subgroups. **[5]**

(iii) Explain why G must be abelian. **[1]**

The group $\mathbb{Z}_k$ is the cyclic group of order k, consisting of the elements $\{0, 1, 2, \ldots, k-1\}$ under the operation $+_k$ of addition modulo k.

The coordinate group C_{mn} is the group which consists of elements of the form (x, y), where $x \in \mathbb{Z}_m$ and $y \in \mathbb{Z}_n$, under the operation $\oplus$ given by $(x_1, y_1)\oplus(x_2, y_2) = (x_1 +_m x_2, y_1 +_n y_2)$. For example, for $m = 5$ and $n = 2$, $(3,0)\oplus(4,1) = (2,1)$.

(b) (i) List all the elements of $J = C_{34}$. **[1]**

(ii) Show that G and J are isomorphic. **[1]**

There is a second coordinate group of order 12; that is, $K = C_{mn}$, where $1 < m < n < 12$ but neither m nor n is equal to 3 or 4.

(c) (i) State the values of m and n which give K. **[1]**

(ii) Hence list all of the elements of K. **[1]**

(iii) Explain why K must be abelian. **[1]**

(d) Show that G and K are **not** isomorphic. **[2]**

END OF QUESTION PAPER