OCR
Oxford Cambridge and RSA

Wednesday 22 May 2024 – Afternoon

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Time allowed: 2 hours 40 minutes

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You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Section A (36 marks)

1 By expressing $\frac{1}{r+1}-\frac{1}{r+2}$ as a single fraction, find $\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}$ in terms of n. **[4]**

2 Two complex numbers are given by $u=-1+\mathrm{i}$ and $v=-2-\mathrm{i}$.

(a) **(i)** Find $u-v$ in the form $a+b\mathrm{i}$, where a and b are real. **[1]**

(ii) **In this question you must show detailed reasoning.**

Find $\frac{u}{v}$ in the form $a+b\mathrm{i}$, where a and b are real. **[3]**

(b) Express u in exact modulus-argument form. **[3]**

3 The equation $2x^3-2x^2+8x-15=0$ has roots α, β and γ.

Determine the value of $\alpha^2+\beta^2+\gamma^2$. **[4]**

4 The equation of a curve is $y=\frac{1}{\sqrt{k^2+x^2}}$, where k is a positive constant. The region between the x-axis, the y-axis and the line $x=k$ is rotated through 2π radians about the x-axis.

Given that the volume of the solid of revolution formed is 1 unit^3, find the exact value of k. **[4]**

5 **(a)** Given that $\mathbf{u}=\begin{pmatrix}-2\\1\\2\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}a\\0\\1\end{pmatrix}$ and $\mathbf{u}\times\mathbf{v}=\begin{pmatrix}1\\b\\3\end{pmatrix}$, find a and b. **[3]**

(b) Using $\mathbf{u}\times\mathbf{v}$, determine the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$, given that this angle is acute. **[3]**

6 On separate Argand diagrams, sketch the set of points represented by each of the following.

(a) $|z-1-2\mathrm{i}| \leqslant 4$. **[3]**

(b) $\arg(z+\mathrm{i}) = \frac{1}{3}\pi$. **[3]**

7 **(a)** Explain why $\int_1^2 \frac{1}{\sqrt[3]{x-2}}\,\mathrm{d}x$ is an improper integral. **[1]**

(b) **In this question you must show detailed reasoning.**

Use an appropriate limit argument to evaluate this integral. **[4]**

Section B (108 marks)

8 **(a)** Specify fully the transformation T of the plane associated with the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and λ is a non-zero constant. **[2]**

(b) **(i)** Find $\det \mathbf{M}$. **[1]**

(ii) Deduce **two** properties of the transformation T from the value of $\det \mathbf{M}$. **[2]**

(c) Prove that $\mathbf{M}^n = \begin{pmatrix} 1 & n\lambda \\ 0 & 1 \end{pmatrix}$, where n is a positive integer. **[4]**

(d) Hence specify fully a **single** transformation which is equivalent to n applications of the transformation T. **[1]**

9 A curve has polar equation $r = a\sin 3\theta$, for $0 \leqslant \theta \leqslant \pi$, where a is a positive constant.

(a) Sketch the curve. Indicate the parts of the curve where r is negative by using a broken line. **[3]**

(b) **In this question you must show detailed reasoning.**

Determine the area of one of the loops of the curve. **[5]**

10 **(a)** Write down the first three terms of the Maclaurin series for $\ln(1+x^3)$. **[1]**

(b) Use these three terms to show that $\ln(1.125) \approx \frac{n}{1536}$, where n is an integer to be determined. **[3]**

(c) Charlie uses the same first three terms of the series to approximate $\ln 9$ and gets an answer of 147, correct to 3 significant figures. However, $\ln 9 = 2.20$ correct to 3 significant figures.

Explain Charlie's error. **[2]**

11 The plane Π has equation $2x-y+2z=4$. The point P has coordinates (8, 4, 5).

(a) Calculate the shortest distance from P to Π. **[2]**

The line L has equation $\frac{x-2}{3}=\frac{y}{2}=\frac{z+3}{4}$.

(b) Verify that P lies on L. **[2]**

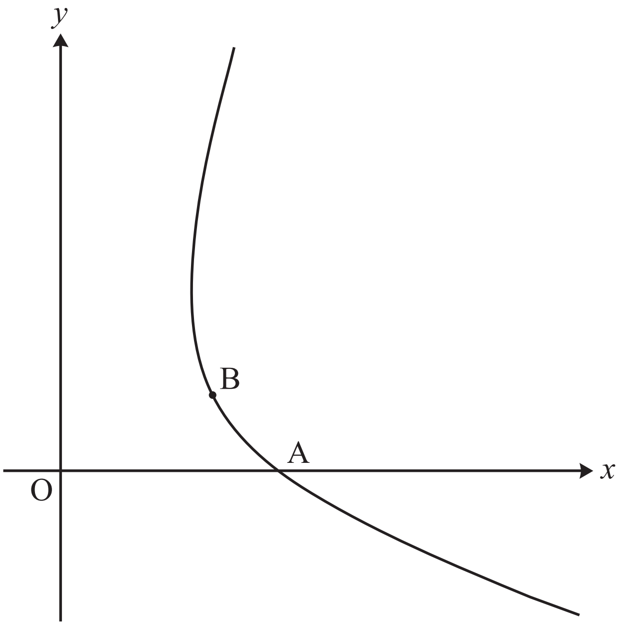
(c) Find the coordinates of the point of intersection of L and Π. **[3]**

(d) Determine the acute angle between L and Π. **[4]**

(e) Use the results of parts **(b)**, **(c)** and **(d)** to verify your answer to part **(a)**. **[3]**

12 The diagram shows the curve with parametric equations

$x=2\cosh t+\sinh t,\ y=\cosh t-2\sinh t$.



(a) The curve crosses the positive x-axis at A.

(i) Determine the value of the parameter t at A, giving your answer in logarithmic form. **[4]**

(ii) Find the x-coordinate of A, giving your answer correct to **3** significant figures. **[2]**

(b) The point B has parameter $t=0$.

Determine the equation of the tangent to the curve at B. **[6]**

13 The complex number z is defined as $z = \frac{1}{3}e^{i\theta}$ where $0 < \theta < \frac{1}{2}\pi$.

On an Argand diagram, the point O represents the complex number 0, and the points $P_1, P_2, P_3, \ldots$ represent the complex numbers $z, z^2, z^3, \ldots$ respectively.

(a) Write down each of the following.

(i) The ratio of the lengths $OP_{n+1} : OP_n$ **[1]**

(ii) The angle $P_{n+1}OP_n$ **[1]**

(b) **(i)** Show that $(3 - e^{i\theta})(3 - e^{-i\theta}) = a + b\cos\theta$, where a and b are integers to be determined. **[2]**

(ii) By considering the sum to infinity of the series $z + z^2 + z^3 + \ldots$, show that

$$\frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \frac{1}{27}\sin 3\theta + \ldots = \frac{3\sin\theta}{10 - 6\cos\theta}.$$ **[6]**

14 **(a)** Find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 12e^{-x}$. **[7]**

You are given that y tends to zero as x tends to infinity, and that $\frac{dy}{dx} = 0$ when $x = 0$.

(b) Find the exact value of x for which $y = 0$. **[5]**

15 Three planes have equations

$$\begin{aligned} x + ky + 3z &= 1, \\ 3x + 4y + 2z &= 3, \\ x + 3y - z &= -k, \end{aligned}$$

where k is a constant.

(a) Show that the planes meet at a point except for one value of k, which should be determined. **[4]**

(b) Show that, when the planes do meet at a point, the y-coordinate of this point is independent of k. **[6]**

16 In this question you must show detailed reasoning.

Show that $\int_0^1 \frac{1}{\sqrt{x^2+x+1}}\,\mathrm{d}x = \ln\left(\frac{a+b\sqrt{3}}{c}\right)$, where a, b and c are integers to be determined. **[6]**

17 In an industrial process, a container initially contains 1000 litres of liquid. Liquid is drawn from the bottom of the container at a rate of 5 litres per minute. At the same time, salt is added to the top of the container at a constant rate of 10 grams per minute. After t minutes the mass of salt in the container is x grams, and you are given that $x = 0$ when $t = 0$.

In modelling the situation, it is assumed that the salt dissolves instantly and uniformly in the liquid, and that adding the salt does not change the volume of the liquid.

(a) (i) Show that the concentration of salt in the liquid after t minutes is $\frac{x}{1000-5t}$ grams per litre. **[1]**

(ii) Hence show that the mass of salt in the container is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{200-t} = 10.$$ **[3]**

(b) Show by integration that $x = 10(200-t)\ln\left(\frac{200}{200-t}\right)$. **[8]**

(c) (i) Hence determine the mass of salt in the container when half the liquid is drawn off. **[2]**

(ii) Determine also the time at which the mass of salt in the container is greatest. **[5]**

(d) When the process is run, it is found that the concentration of salt over time is higher than predicted by the model.

Suggest a reason for this. **[1]**

END OF QUESTION PAPER