OCR
Oxford Cambridge and RSA

Monday 24 June 2024 – Afternoon

A Level Further Mathematics B (MEI)

Y436/01 Further Pure with Technology

Time allowed: 1 hour 45 minutes

1419044815

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a computer with appropriate software
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

1 A family of curves is given by the equation

$$y = \frac{x^2 - x + a^2 - a}{x - 1} \quad (*)$$

where the parameter a is a real number.

(a) **(i)** On the axes in the Printed Answer Booklet, sketch the curve in each of these cases.

- $a = -0.5$
- $a = -0.1$
- $a = 0.5$ **[3]**

(ii) State **one** feature of the curve for the cases $a = -0.5$ and $a = -0.1$ that is **not** a feature of the curve in the case $a = 0.5$. **[1]**

(iii) By using a slider for a, or otherwise, write down the **non-zero** value of a for which the points on the curve (*) all lie on a straight line. **[1]**

(iv) Write down the equation of the vertical asymptote of the curve (*). **[1]**

The equation of the curve (*) can be written in the form $y = x + A + \frac{a^2 - a}{x - 1}$, where A is a constant.

(v) Show that $A = 0$. **[2]**

(vi) Hence, or otherwise, find the value of

$$\lim_{x \to \infty}\left(\frac{x^2 - x + a^2 - a}{x - 1} - x\right).$$ **[2]**

(vii) Explain the significance of the result in part **(a)(vi)** in terms of a feature of the curve (*). **[1]**

(b) In this part of the question the value of the parameter a satisfies $0 < a < 1$. For values of a in this range the curve intersects the x-axis at points X and Y. The point Z has coordinates $(0, -1)$. These three points form a triangle XYZ.

(i) Determine, in terms of a, the area of the triangle XYZ. **[4]**

(ii) Find the maximum area of the triangle XYZ. **[2]**

2 Wilson's theorem states that a positive integer $n > 1$ is prime if and only if $(n-1)! \equiv n-1 \pmod{n}$.

(a) (i) Calculate $136! \pmod{137}$. **[1]**

(ii) Hence, determine if the integer 137 is prime. **[1]**

(b) (i) Create a program that uses Wilson's theorem to find all prime numbers less than or equal to n.
Write down your program in the Printed Answer Booklet. **[3]**

(ii) Using part **(b)(i)**, write down **all** prime numbers x, where $260 \leqslant x \leqslant 300$. **[1]**

(c) (i) Explain why there is exactly **one** prime number congruent to $2 \pmod{4}$. **[1]**

(ii) Explain why **no** prime number is congruent to $0 \pmod{4}$. **[1]**

(iii) Using part **(b)(ii)**, write down the **three** prime numbers y, where $260 \leqslant y \leqslant 300$, that are congruent to $3 \pmod{4}$. **[1]**

Label the three prime numbers in part **(c)(iii)** c_1, c_2 and c_3. Define the integer N by

$N = 4c_1c_2c_3 + 3$.

(iv) Explain why N is **not** divisible by c_1, c_2 or c_3. **[1]**

(v) Write down the value of $N \pmod{4}$. **[1]**

(d) The fundamental theorem of arithmetic states that every positive integer can be written uniquely as the product of powers of prime numbers.

Suppose there are finitely many prime numbers congruent to $3 \pmod{4}$. Label these prime numbers $p_1, \ldots, p_n$, where $p_1 = 3$. Using the $n-1$ integers $p_2, \ldots, p_n$, define the integer M by

$M = 4p_2p_3 \ldots p_n + 3$.

(i) Write down the value of $M \pmod{4}$. **[1]**

(ii) Explain why $M = q_1q_2 \ldots q_r$, where the integers $q_1, \ldots, q_r$ are all prime. **[2]**

(iii) Prove that there is at least one integer i, where $1 \leqslant i \leqslant n$, such that $q_i \equiv 3 \pmod{4}$. **[4]**

(iv) Hence, deduce that there are infinitely many prime numbers congruent to $3 \pmod{4}$. **[5]**

3 This question concerns the family of differential equations

$$\frac{dy}{dx} = x^2 - y + a\cos(x)\cos(y) \quad (**)$$

where a is a constant, $x \geqslant 0$ and $y > 0$.

(a) In this part of the question $a = 0$.

(i) Find the solution to (**) in which $y = 1$ when $x = 0$. **[1]**

(ii) In this part of the question m is a real number. Show that the equation of the isocline $\frac{dy}{dx} = m$ is a parabola. **[1]**

(iii) Using the result given in part **(a)(ii)**, or otherwise, sketch the tangent field for (**) on the axes in the Printed Answer Booklet. **[2]**

(b) **Fig. 3.1** and **Fig. 3.2** show the tangent fields for two distinct and unspecified values of a. In each case, a sketch of the solution curve $y = g(x)$ which passes through the point (0, 2) is shown for $0 \leqslant x \leqslant \frac{1}{2}$.

Fig. 3.1

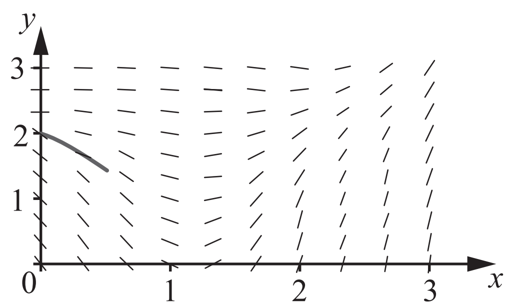
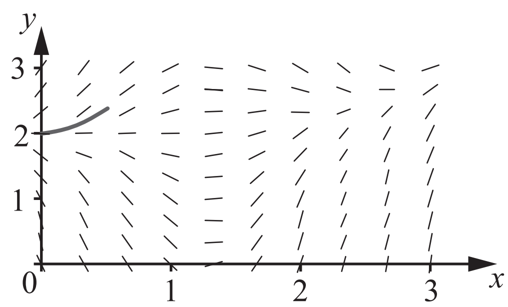


Fig. 3.2



(i) In each case, continue the sketch of the solution curve for $\frac{1}{2} \leqslant x \leqslant 3$ on the axes in the Printed Answer Booklet. **[2]**

(ii) State **one** feature which is present in the continued solution curve for **Fig. 3.1** that is **not** a feature of the continued solution curve for **Fig. 3.2**. **[1]**

(iii) Using a slider for a, or otherwise, estimate the value of a for the solution curve shown in **Fig. 3.2**. **[1]**

(c) The Euler method for the solution of the differential equation $\frac{dy}{dx} = f(x, y)$ is as follows.

$$y_{n+1} = y_n + hf(x_n, y_n)$$
$$x_{n+1} = x_n + h$$

(i) Construct a spreadsheet to solve (**), so that the value of a and the value of h can be varied, in the case $x_0 = 0$ and $y_0 = 1$. State the formulae you have used in your spreadsheet. **[2]**

(ii) In this part of the question $a = 0$. Use your spreadsheet with $h = 0.1$ to approximate the value of y when $x = 0.5$ for the solution to (**) in which $y = 1$ when $x = 0$. **[1]**

(iii) Using part **(a)(i)**, state the accuracy of the approximate value of y given in part **(c)(ii)**. **[1]**

(iv) State **one** change to your spreadsheet that could improve the accuracy of the approximate value of y found in part **(c)(ii)**. **[1]**

(d) The modified Euler method for the solution of the differential equation $\frac{dy}{dx} = f(x, y)$ is as follows.

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + h, y_n + k_1)$$
$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$
$$x_{n+1} = x_n + h$$

(i) Adapt your spreadsheet from part **(c)(i)** to a spreadsheet to solve (**), so that the value of a and the value of h can be varied, in the case $x_0 = 0$ and $y_0 = 1$. State the formulae you have used in your spreadsheet. **[2]**

(ii) In this part of the question $a = -0.5$. Use the spreadsheet from part **(d)(i)** with $h = 0.1$ to approximate the value of y when $x = 0.5$ for the solution to (**) in which $y = 1$ when $x = 0$. **[1]**

In this part of the question $a = -0.5$. The solution to (**) in which $y = 1$ when $x = 0$ has a turning point with coordinates (c, d) where $0 < c < 1$.

(iii) Use the spreadsheet in part **(d)(i)** to determine the value of c correct to **1** decimal place. **[2]**

(iv) Use the spreadsheet in part **(d)(i)** to determine the value of d correct to **3** decimal places. **[2]**

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